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APPLICATION OF THE VIBRATION ANALYSIS OF LINEAR SYSTEMS WITH TIME-PERIODIC COEFFICIENTS TO THE DYNAMICS OF A ROLLING TYRE

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In this article, theories of rolling tyre vibrations are presented. In previous publications, tread pattern was neglected and authors have studied vibrations in smooth tyres. When heterogeneity caused by a tread pattern on the tyre belt is introduced, it is shown that vibrations can be described by linear equations with time periodic coefficients. Firstly, the perturbation method is applied for a nearly smooth tyre, and the "self-excitation" phenomenon, a general feature in time periodic linear systems, is illustrated with the semi-analytic expressions obtained. Then, the generalization to a strong heterogeneity is achieved using the Bloch wave theory. This theoretical background suggests the decomposition of experimental data of noise in time signals for a given phase as compared to the wheel rotation. Finally, an effective method for numerical computations of vibrations is proposed; it uses the Floquet theory, a consequence of the Bloch theory. Finite element formulation and algorithm are derived for the heterogeneous "circular ring model".

1. INTRODUCTION

The propagation of waves in tyres is a cause of rolling noise. Compared to classic vibration analysis, rolling tyres have specific characteristics that make the results of modal analysis differ from experimental data.

The first characteristic of tyres is that they roll on road. The classic modal analysis is developed for vibration analysis of a nearly immobile object, when the particles in contact with other objects remain the same. In the case of tyres, the structure has a global rotation, and the particles in contact with the road change with time.

Some results on the vibration analysis of a rolling tyre were achieved in the case of smooth tyres. In Huang [1] or in Vinesse and Nicollet [2], the rotation was introduced in vibration analysis. In reference [1] the contact with the road has also been investigated. The main problem was to take into account a contact condition

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with the road. The method used by these authors is called impedance method and consists in solving the following loop problem. At first, the contact with the road is represented by given loads on the contact point; these loads are assumed to be at constant frequency. The vibration solution of the rolling tyre submitted to these loads is calculated and the solution in the contact point with the road is examined. Then, the loads that represent the road action are modified until a zero displacement in the contact point is obtained. This problem has solutions only for some frequencies. It is shown that these frequencies are the natural frequencies of a rolling tyre, with contact. These studies showed that both mode shapes and natural frequencies depend on rolling speed.

The second characteristic of commercial tyres is that they are not smooth. They have a tread pattern and this pattern turns when tyres roll.

Impulse excitations at different instants t_1 and t_2 of the same tyre give different measures of pressure after a delay τ, depending on the tread pattern position comparative to the ground: if rubber blocks are not in the same position as compared to the ground at time t_1 and t_2, then vibrations and sound measured at time $(t_1 + \tau)$ and $(t_2 + \tau)$ will be different. So, it is shown that there exists an influence of the rotation of the rubber blocks on the dynamic properties of a rolling tyre. The presence of grooves in the tread makes frequency uncoupling impossible, and modal analysis does not apply to this kind of systems.

Note that a model of a non-rolling smooth tyre was used for the study of wave propagation in a rolling tyre with a tread pattern by Kung *et al.* [3] (and related articles [4–6]). The effect of the moving tread pattern on wave propagation was neglected in these articles.

The aim of this article is to present the vibration theories that allows one to take the tread pattern into account and to compare theoretical results with experimental analysis.

For simplification, equilibrium equations obtained for a “circular ring model” will be used. They have been recently used by Huang [1], Kropp [7] or Cuschieri *et al.* [8], and they are reiterated in section 2.

When the belt heterogeneity is small, the tyre can be considered as nearly smooth. Starting from the modal decomposition of vibrations in a smooth tyre studied in references [1] or [4], the perturbation theory is applied in section 3 to take into account a weak tread pattern effect. In this work, it is shown that the difference from modal analysis is that when the structure is excited at pulsation ω_0 by an external force, the response contains all pulsations $\omega_0 + q\tilde{\Omega}$, where q is a relative integer and $\tilde{\Omega}$ is the fundamental pulsation of tread pattern impact on the road. This is caused by internal forces of disequilibrium usually called “self-excitation forces”.

Nevertheless, this theory is approximate and can be applied only when internal forces of disequilibrium are weak. This theory may not be suitable for tyres with a strong heterogeneity (for example tyres with a winter tread pattern), or at high rolling speed. Linear systems with time periodic coefficients are generally characterized by Bloch waves. In section 4, the theory of Bloch waves will be explained and the consequences on experimental procedures will be derived. The decomposition into Bloch waves is similar to modal decomposition and the Bloch wave presentation of experimental or numerical data makes the dynamics of a rolling tyre easy to understand.

Nevertheless, the Bloch wave decomposition is not of practical interest for vibration computations, because it requires the diagonalization of very large matrices whose sizes are the number of degrees of freedom of the mechanical model, times the number of time steps in one period of revolution of the wheel. At high rolling speed, where the spectrum contains high frequencies, the time steps can be very small to catch these frequencies. Moreover, the spatial mesh can be very fine to catch the small wavelengths. Because of this, the Bloch wave decomposition is usually not used for computations, and the Floquet theory is preferred.

The Floquet theory is a consequence of the Bloch wave decomposition: the decomposition in Bloch waves is partially predicted with the Floquet theory. The resolution is based on the calculus of the transition matrix. Its size is just the double of the number of degrees of freedom in the mechanical model. The transition matrix diagonalization exhibits the Floquet coefficients. These coefficients show the influence of tyre characteristics (for example damping) on vibration properties, like the natural frequencies in the classical theory, and allow the qualitative study of the dynamic response, especially if the excitation sources are unknown. In Bradley [9], this theory was used for the study of the propagation of waves in ducts with a periodic cross-section. In Sinha and Butcher [10] and in Guttalu and Flashner [11], this theory was used for the study of the stability of heterogeneous objects in rotation. No external excitation was considered. It will be seen in section 5 that the method can be extended to the present case with excitation and equations for tyres will be derived.

In the main body of this article, theories when the damping is neglected will be presented. In the Appendices, damping is introduced.

2. EQUILIBRIUM EQUATIONS OF THE "CIRCULAR RING MODEL"

A polar co-ordinate system (r, θ) is used. It is centred on the centre of the wheel. The radial vector is called $\mathbf{e}_r$, the tangential vector is called $\mathbf{e}_\theta$.

The tyre belt is modelled by a circular beam of radius a and of width z, turning at constant rotational speed Ω in a plane in translation with the car. The position of a particle at time 0 is indexed by β, its polar angle on the circle of radius a. Its position at time t is indexed by the angle $\theta = \beta + \Omega t$. The value of this position is the sum of $\mathbf{X}(t, \theta)$, the position obtained when inertia and viscosity effects are neglected, and a displacement that allows the reintroduction of dynamics and corrects the first approximation $\mathbf{X}(t, \theta)$. This displacement is denoted by $u_r(t, \theta)$ in the radial direction and by $u_\theta(t, \theta)$ in the tangential direction (see Figure 1).

Only equilibrium equations satisfied by u_r and u_θ are studied, and the value $\mathbf{X}(t, \theta)$ is assumed as known. Its influence on the displacement is a force with components q_r and q_θ (see Campanac [12]).

2.1. DERIVATION OF THE LAGRANGIAN OF THE SYSTEM

Recall briefly the mechanical hypothesis of the "circular ring model" of a rolling tyre. The method to obtain equilibrium equations is the same as the one used the

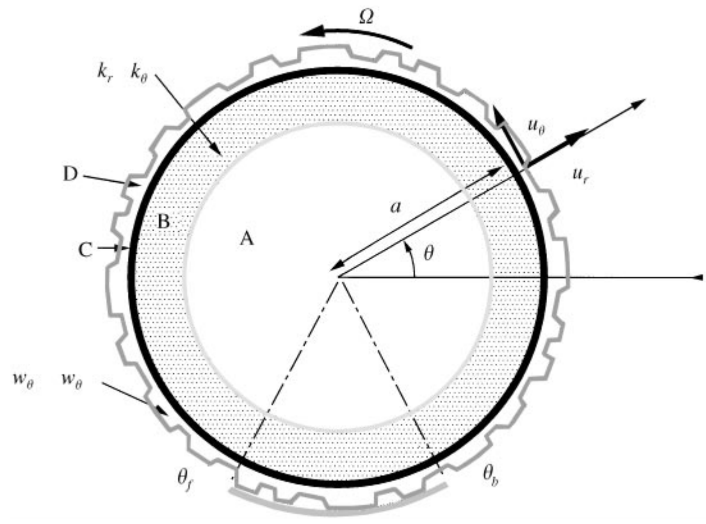

Figure 1. The tyre geometry. A: rim; B: sidewall; C: carcass; D: tread.

cited articles, that is, derive kinetic energy, elastic energy and potential energy for the system, and use Lagrange principle to get equilibrium equations.

The main difference from the former publications are:

1. The referential is chosen in translation with the car but not in rotation with the wheel. This referential is more natural for the study of rolling tyre vibrations.
2. A differential system of first order in time is needed for the natural frequency problem. The displacement vector field $\mathbf{u}(t, \theta)$ and the relative speed vector field $\mathbf{v}(t, \theta)$ are used in the formulation of the problem.

2.1.1. *The kinetic energy T*

The displacement of a turning particle that is in $\theta = \theta_0$ at time $t = t_0$, is given by

$$u_r(t, \theta_h)\mathbf{e}_r(\theta_h) + u_\theta(t, \theta_h)\mathbf{e}_\theta(\theta_h) \tag{1}$$

where $\theta_h = \theta_0 + \Omega(t - t_0)$. The speed of this particle is the total derivative in t. The derivative with respect to time of u_r (respectively u_θ) is denoted by $\dot{u}_r$ (respectively $\dot{u}_\theta$), the derivative with respect to θ of u_r (respectively u_θ) is denoted by u_r' (respectively u_θ'). The speed vector is then

$$\begin{aligned} v_r &= \dot{u}_r + \Omega u_r' - \Omega u_\theta, \\ v_\theta &= \dot{u}_\theta + \Omega u_\theta' + \Omega u_r. \end{aligned} \tag{2}$$

The kinetic energy by unit of length T is then

$$T(v_r, v_\theta) = \tfrac{1}{2}\,\rho({v_r}^2 + {v_\theta}^2) \tag{3}$$

where ρ is the density of the beam, constant in smooth tyres.

This expression is equivalent to the one in the cited articles. The difference comes from the different referential used here, that is in translation with the car but not in rotation with the wheel.

2.1.2. *The deformations of the system*

The tangential transformation at a given time t is calculated:

$$u_r(t, \theta + \mathrm{d}\theta)\mathbf{e}_r(t, \theta + \mathrm{d}\theta) + u_\theta(t, \theta + \mathrm{d}\theta)\mathbf{e}_\theta(t, \theta + \mathrm{d}\theta) - u_r(t, \theta)\mathbf{e}_r(t, \theta) - u_\theta(t, \theta)\mathbf{e}_\theta(t, \theta) = \left(\frac{1}{a}u_r + \frac{1}{a}u'_\theta\right)a\mathrm{d}\theta\,\mathbf{e}_\theta + \left(\frac{1}{a}u_\theta - \frac{1}{a}u'_r\right)(-a\mathrm{d}\theta\,\mathbf{e}_r). \tag{4}$$

During the transformation, an elementary vector $a\mathrm{d}\theta\mathbf{e}_\theta$ is extended by longitudinal strain ε_θ. Looking at the first term of the tangential transformation, the value of the deformation is found to be

$$\varepsilon_\theta = \frac{1}{a}u_r + \frac{1}{a}u'_\theta. \tag{5}$$

The elementary vector is also rotated around the direction perpendicular to the plane $\mathbf{e}_z$ by an angle γ_z. Remembering that $\mathbf{e}_z \wedge (a\mathrm{d}\theta\,\mathbf{e}_\theta) = (-a\mathrm{d}\theta\,\mathbf{e}_r)$, it is found that rotation is the second term of the equation. The value of the rotation is

$$\gamma_z = \frac{1}{a}u_\theta - \frac{1}{a}u'_r. \tag{6}$$

The bending is by definition the derivative with respect to the co-ordinate $a\theta$ of the rotation of the normal vector to the beam. It is assumed that the rotation of the normal vector to beam is the same as the rotation of the elementary vector, and so, the bending $\chi_\theta(t, \theta)$ is

$$\chi_\theta = \frac{1}{a}\frac{\partial\gamma_z}{\partial\theta} = \frac{1}{a^2}u'_\theta - \frac{1}{a^2}u''_r. \tag{7}$$

2.1.3. *The potential energy V*

External forces are centrifugal force, inflation pressure, force of excitation by road bumps, and force of tread pattern excitation in the case of commercial tyres. Centrifugal force is a radial force whose value is $\rho\Omega^2 a$. Pressure is also a radial force, its value being $p_0 z$ where p_0 is the value of the inflation pressure and z is the width of the tyre. Excitation by road bumps and by tread pattern are modelled by a force (see reference [12]). This force has two components $q_r(t, \theta)$ and $q_\theta(t, \theta)$.

Finally, the potential energy is:

$$V(u_r, u_\theta) = -q_r u_r - q_\theta u_\theta - (p_0 z + \rho\Omega^2 a)u_r. \tag{8}$$

2.1.4. *Initial tension*

Centrifugal force and pressure are balanced by the initial tension in the tyre t_i. The equilibrium equation of a segment of length $a\delta\theta$ gives

$$t_i(\mathbf{e}_\theta(\theta + \delta\theta) - \mathbf{e}_\theta(\theta)) + (\rho\Omega^2 a + p_0 z)\mathbf{e}_r a\delta\theta = 0. \tag{9}$$

This solution neglects the action of the sidewall, the reaction of the road, and the geometry changes. It is therefore approximate. Finally, the value found for t_i is

$$t_i = a^2\rho\Omega^2 + azp_0. \tag{10}$$

2.1.5. *The elastic energy U*

Although it is not always the case in layered beams, the elastic energy per unit of length is assumed to be uncoupled in a potential of free energy of longitudinal strain and a potential of free energy of bending:

$$U_e(u_r, u_r', u_r'', u_\theta, u_\theta') = \psi_r(\varepsilon_\theta, \gamma_z) + \psi_b(\chi_\theta). \tag{11}$$

The bending energy is a quadratic function of χ_θ:

$$\psi_b(\chi_\theta) = \tfrac{1}{2} D\chi_\theta^2, \tag{12}$$

where D is a bending stiffness of a beam. It is a constant in smooth tyres.

The potential of free energy of strain is a polynomial of the Cauchy strain e:

$$e = \tfrac{1}{2}((1 + \varepsilon_\theta)^2 + \gamma_z{}^2 - 1). \tag{13}$$

It has a linear term caused by initial tension:

$$\psi_r(e) = \psi_r(0) + t_i e + \tfrac{1}{2} K(e)^2, \tag{14}$$

where K is a beam stiffness, constant in smooth tyres.

This expression is linearized up to second order:

$$\psi_r(e) \approx \psi_r(0) + t_i\varepsilon_\theta + \tfrac{1}{2} t_i\gamma_z^2 + \tfrac{1}{2}(t_i + K)\varepsilon_\theta^2. \tag{15}$$

In this expression the term t_i is considered negligible in comparison with the term K:

$$\psi_r(e) \approx \psi_r(0) + t_i\varepsilon_\theta + \tfrac{1}{2}\, t_i\gamma_z^2 + \tfrac{1}{2}\, K\varepsilon_\theta^2 . \tag{16}$$

The sidewalls are modelled by an elastic linear force that is applied to the belt. The elastic behavior is assumed to be uncoupled in the radial and tangential directions, and therefore the force of the sidewalls is characterized by two stiffnesses, k_r in the radial direction and k_θ in the tangential direction:

$$U_s(u_r, u_\theta) = \tfrac{1}{2}\, k_r u_r^2 + \tfrac{1}{2}\, k_\theta u_\theta^2 . \tag{17}$$

In Schramm [13] and Hamet [14], it was suggested that the part of the tread in contact with the road has an elastic energy. It is assumed that the contact area is delimited by two given angles θ_f for the front of the contact area, and θ_b for the back of the contact area, and that the tread is characterized by two stiffnesses, an axial stiffness w_r and a shear stiffness w_θ:

$$\forall\ \theta \in [\theta_f, \theta_b] \quad U_{tp}(u_r, u_\theta) = \tfrac{1}{2}\, w_r u_r^2 + \tfrac{1}{2}\, w_\theta {u_\theta}^2 . \tag{18}$$

For smooth tyres, these two stiffnesses are functions of θ, and are zero outside the contact area $[\theta_f, \theta_b]$.

2.2. THE EQUATIONS OF MOTION

In the following, equations involving only derivative of the first order in time are required. Because of this, the two components of relative speed will be used as an intermediate unknown. The two equilibrium equations and the two equations defining the relative speed are regrouped into a four-equation system.

The next two equations define the speed vector value. For convenience, they are multiplied by the density ρ:

$$\begin{aligned} \rho v_r &= \rho(\dot{u}_r + \Omega u_r' - \Omega u_\theta), \\ \rho v_\theta &= \rho(\dot{u}_\theta + \Omega u_\theta' + \Omega u_r). \end{aligned} \tag{19}$$

Then, v_r and v_θ are replaced by their values in the expression of the kinetic energy T. The Lagrangian density l is derived, equilibrium equations are obtained for a stationary point of the Lagrangian L:

$$\begin{aligned} l(u_r, u_r', u_r'', \dot{u}_r, u_\theta, u_\theta', \dot{u}_\theta) &= T(u_r, u_r', \dot{u}_r, u_\theta, u_\theta', \dot{u}_\theta) - [U_e + U_s + U_{tp} + V], \\ L &= \int_t \int_{-\pi}^{\pi} l(u_r, u_r', u_r'', \dot{u}_r, u_\theta, u_\theta', \dot{u}_\theta)\, a\, \mathrm{d}\theta\, \mathrm{d}t. \end{aligned} \tag{20}$$

The derivative with respect to u_r gives the first equilibrium equation. Substituting on the left-side the expression of acceleration and on the right-hand side the terms of internal and external forces, one has:

$$\frac{\partial l}{\partial u_r} - \left(\frac{\partial l}{\partial u'_r}\right)' - \overbrace{\left(\frac{\partial l}{\partial \dot{u}_r}\right)}^{\cdot} + \left(\frac{\partial l}{\partial u''_r}\right)'' = 0,$$

$$\rho(\dot{v}_r + \Omega v'_r - \Omega v_\theta) + (\dot{\rho} + \Omega\rho')v_r = \frac{1}{a^2}(D\chi_\theta)'' - \frac{1}{a}K\varepsilon_\theta - \frac{1}{a}t_i\gamma'_z - (k_r + w_r)u_r + q_r. \tag{21}$$

The derivative with respect to u_θ gives the second equilibrium equation:

$$\frac{\partial l}{\partial u_\theta} - \left(\frac{\partial l}{\partial u'_\theta}\right)' - \overbrace{\left(\frac{\partial l}{\partial \dot{u}_\theta}\right)}^{\cdot} = 0,$$

$$\rho(\dot{v}_\theta + \Omega v'_\theta + \Omega v_r) + (\dot{\rho} + \Omega\rho')v_\theta = \frac{1}{a^2}(D\chi_\theta)' + \frac{1}{a}(K\varepsilon_\theta)' - \frac{1}{a}t_i\gamma_z - (k_\theta + w_\theta)u_\theta + q_\theta. \tag{22}$$

In equations (21, 22), the terms in $(\dot{\rho} + \Omega\rho')$ are zero because ρ is a constant. The system is then

$$-\frac{1}{a^2}(D\chi_\theta)'' + \frac{1}{a}K\varepsilon_\theta + \frac{1}{a}t_i\gamma'_z + (k_r + w_r)u_r + \rho(\dot{v}_r + \Omega v'_r - \Omega v_\theta) = q_r,$$

$$-\frac{1}{a^2}(D\chi_\theta)' - \frac{1}{a}(K\varepsilon_\theta)' + \frac{1}{a}t_i\gamma_z + (k_\theta + w_\theta)u_\theta + \rho(\dot{v}_\theta + \Omega v'_\theta + \Omega v_r) = q_\theta,$$

$$-\rho(\dot{u}_r + \Omega u'_r - \Omega u_\theta) + \rho v_r = 0, \tag{23}$$

$$-\rho(\dot{u}_\theta + \Omega u'_\theta + \Omega u_r) + \rho v_\theta = 0.$$

In this system ρ, D, K are constants. Later, a dependence in θ and t will be introduced.

Note that the form of the equations is also valid for heterogeneous mechanical properties. The terms in $(\dot{\rho} + \Omega\rho')$ is always zero. The physical reason is just the mass conservation that leads to

$$\dot{\rho} + \Omega\rho' = 0. \tag{24}$$

3. THE APPRO XIMATE EFFECT OF THE TREAD PATTERN

In this section, it is assumed that the tyre can be considered at first sight as smooth. A first approximation of the solution is calculated with this hypothesis, using the modal decomposition in a smooth tyre.

Then, a small heterogeneity is taken into account. When the approximate solution is introduced in the model with heterogeneity, it is shown that it is not in equilibrium. A correction that partially takes into account these disequilibrium forces is calculated.

3.1. MODAL DECOMPOSITION FOR A SMOOTH ROLLING TYRE

Recall first the modal decomposition method for a smooth rolling tyre. The method presented here is a direct formulation, different from the impedance method. It will be used in the next section in the calculation of the first approximation.

3.1.1. *The natural frequencies*

For a fixed value ω of the pulsation, it is possible to find a solution $u_r(t,\theta) = u_r(\omega,\theta)\exp(\mathrm{i}\omega t)$ (and the same form for u_θ, v_r, v_θ). Introducing this form in equilibrium equations (23) without the source term, after simplification by $\exp(\mathrm{i}\omega t)$ and rearrangement, equation (23) leads to the following system:

$$-\frac{1}{a^2}(D\chi_\theta)'' + \frac{1}{a}K\varepsilon_\theta + \frac{1}{a}t_i\gamma_z' + (k_r + w_r)u_r + \Omega\rho(v_r' - v_\theta) = i\omega(-\rho)v_r,$$

$$-\frac{1}{a^2}(D\chi_\theta)' - \frac{1}{a}(K\varepsilon_\theta)' + \frac{1}{a}t_i\gamma_z + (k_\theta + w_\theta)u_\theta + \Omega\rho(v_\theta' + v_r) = i\omega(-\rho)v_\theta,$$

$$-\Omega\rho(u_r' - u_\theta) + \rho v_r = \mathrm{i}\omega\rho u_r,$$

$$-\Omega\rho(u'_\theta + u_r) + \rho v_\theta = \mathrm{i}\omega\rho u_\theta. \qquad (25)$$

Note that $\mathbf{s}(\omega,\theta)$ is the state vector field. It has four components that are periodic functions in polar angle θ:

$$\mathbf{s}(\omega,\theta) = (u_r(\theta), u_\theta(\theta), v_r(\theta), v_\theta(\theta))^{\mathrm{T}}. \qquad (26)$$

The problem to be solved has the following structure:

$$\mathbf{b}\langle\mathbf{s}\rangle = \mathrm{i}\omega\,\mathbf{m}\cdot\mathbf{s}. \qquad (27)$$

$\mathbf{b}(\theta)$ is a (4×4) matrix of linear differential operators in polar angle with constant coefficients:

$$\mathbf{b}=\begin{pmatrix} \frac{1}{a^4}D\frac{\partial^4}{\partial\theta^4}-\frac{1}{a^2}t_i\frac{\partial^2}{\partial\theta^2}+\frac{1}{a^2}K+k_r+w_r & -\frac{1}{a^4}D\frac{\partial^3}{\partial\theta^3}+\frac{1}{a^2}(t_i+K)\frac{\partial}{\partial\theta} & \Omega\rho\frac{\partial}{\partial\theta} & -\Omega\rho \\ \frac{1}{a^4}D\frac{\partial^3}{\partial\theta^3}-\frac{1}{a^2}(t_i+K)\frac{\partial}{\partial\theta} & -\frac{1}{a^2}\left(K+\frac{1}{a^2}D\right)\frac{\partial^2}{\partial\theta^2}+\frac{1}{a^2}t_i+k_\theta+w_\theta & \Omega\rho & \Omega\rho\frac{\partial}{\partial\theta} \\ -\Omega\rho\frac{\partial}{\partial\theta} & \Omega\rho & \rho & 0 \\ -\Omega\rho & -\Omega\rho\frac{\partial}{\partial\theta} & 0 & \rho \end{pmatrix} \quad (28)$$

$\mathbf{m}$ is a (4×4) constant, antisymmetric matrix:

$$\mathbf{m}=\begin{pmatrix} 0 & 0 & -\rho & 0 \\ 0 & 0 & 0 & -\rho \\ \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \end{pmatrix}. \quad (29)$$

Now examine the weak formulation of the diagonalization problem. Four functions $\delta u_r, \delta u_\theta, \delta v_r, \delta v_\theta$ of $[-\pi,\pi]$, periodic in θ are chosen, and these four components are grouped in the vector $\delta\mathbf{s}$. Note that $\overline{\delta\mathbf{s}}$ is the complex conjugate of the vector $\delta\mathbf{s}$, and $\delta\mathbf{s}^*$ the transposed vector of the complex conjugate of $\delta\mathbf{s}$. $\delta\mathbf{s}^*=\overline{\delta\mathbf{s}}^{\mathrm{T}}$ is also called the adjoint vector. Multiplying the first equation by $\overline{\delta u_r}$, the second by $\overline{\delta u_\theta}$, the third by $\overline{\delta v_r}$ and the fourth by $\overline{\delta v_\theta}$, integrating on $[-\pi,\pi]$, and adding all the equations, equation (27) become

$$\int_{-\pi}^{\pi}\delta\mathbf{s}^*\cdot\mathbf{b}\langle\mathbf{s}\rangle\,\mathrm{d}\theta=\mathrm{i}\omega\int_{-\pi}^{\pi}\delta\mathbf{s}^*\cdot\mathbf{m}\cdot\mathbf{s}\,\mathrm{d}\theta. \quad (30)$$

The symmetry properties of the operator $\mathbf{b}\langle.\rangle$ are obtained when $\delta\mathbf{s}$ is exchanged with $\mathbf{s}$, and simultaneously, the conjugate of t he sesqui-linear forms is taken. Set the elastic form

$$U(\delta s, s) = \int_{-\pi}^{\pi} \overline{\delta u_r}\left(-\frac{1}{a^2}(D\chi_\theta)'' + \frac{1}{a}K\varepsilon_\theta + \frac{1}{a}t_i\gamma_z' + (k_r + w_r)u_r \right) \mathrm{d}\theta + \int_{-\pi}^{\pi} \overline{\delta u_\theta}\left(-\frac{1}{a^2}(D\chi_\theta)' - \frac{1}{a}(K\varepsilon_\theta)' + \frac{1}{a}t_i\gamma_z + (k_\theta + w_\theta)u_\theta \right) \mathrm{d}\theta. \tag{31}$$

Making the usual integration by parts, and using the periodicity properties one has

$$U(\delta \mathbf{s}, \mathbf{s}) = \int_{-\pi}^{\pi} (\overline{\delta\chi_\theta} D\chi_\theta + \overline{\delta\varepsilon_\theta} K\varepsilon_\theta + \overline{\delta\gamma_z} t_i\gamma_z + \overline{\delta u_r}(k_r + w_r)u_r + \overline{\delta u_\theta}(k_\theta + w_\theta)u_\theta)\,\mathrm{d}\theta. \tag{32}$$

Set the gyroscopic form

$$G(\delta \mathbf{s}, \mathbf{s}) = \Omega \int_{-\pi}^{\pi} \rho\,[\overline{\delta u_r}\,.\,(v_r' - v_\theta) + \overline{\delta u_\theta}\,.\,(v_\theta' + v_r)]\,\mathrm{d}\theta - \Omega \int_{-\pi}^{\pi} \rho[\overline{\delta v_r}(u_r' - u_\theta) + \overline{\delta v_\theta}(u_\theta' + u_r)]\,\mathrm{d}\theta. \tag{33}$$

With integration by parts in angle in the second integral, the periodicity of all functions, and the property of the linear mass $\rho' = 0$, are obtains

$$G(\delta \mathbf{s}, \mathbf{s}) = \Omega \int_{-\pi}^{\pi} \rho[\overline{\delta u_r} \cdot (v_r' - v_\theta) + \overline{\delta u_\theta} \cdot (v_\theta' + v_r)]\,\mathrm{d}\theta + \Omega \int_{-\pi}^{\pi} \rho\,[u_r(\overline{\delta v_r' - \delta v_\theta}) + u_\theta(\overline{\delta v_\theta' + \delta v_r})]\,\mathrm{d}\theta. \tag{34}$$

Introducing these formulae, the following result is obtained:

$$\int_{-\pi}^{\pi} \delta \mathbf{s}^* \cdot \mathbf{b}\langle \mathbf{s} \rangle\,\mathrm{d}\theta = U(\delta \mathbf{s}, \mathbf{s}) + G(\delta \mathbf{s}, \mathbf{s}) + \int_{-\pi}^{\pi} (\overline{\delta v_r}\rho v_r + \overline{\delta v_\theta}\rho v_\theta)\,\mathrm{d}\theta. \tag{35}$$

With this expression, $b\langle . \rangle$ is obviously Hermitian.

By definition, matrix $\mathbf{m}$ is antisymmetric, and as a consequence, i$\mathbf{m}$ is also Hermitian.

The numerical solution of this problem with the finite element method is the codiagonalization of two Hermitian matrices. Mode shapes are complex vectors, but it is possible to calculate separately the real and the imaginary parts of these vectors. The method is presented in Geradin and Rixen [15, pp. 76–81]. The natural frequencies and mode shapes depend on the rolling speed Ω because of gyroscopic forces in $\mathbf{b}\langle \cdot \rangle$.

For the discretized system with N degrees of freedom, the state vector has $2N$ degrees of freedom. Operators $\mathbf{b}\langle . \rangle$ and $\mathbf{m}$ are $(2N \times 2N)$ matrices, and problem (27) has in general $2N$ real solutions. These solutions can be separated into two lists of opposite values: taking the conjugate of equation (27), it is found that

$$\mathbf{b}\langle \bar{\mathbf{s}} \rangle = -\mathrm{i}\omega\, \mathbf{m} \cdot \bar{\mathbf{s}}. \tag{36}$$

This means that if ω is a natural frequency and $\mathbf{s}$ is its modal shape function, modal shape function $\bar{\mathbf{s}}$ has a natural frequency $-\omega$. Note that $\omega^1, \omega^2, \ldots, \omega^N$ are positives pulsations. This set, and the set of opposite values are equivalent to N natural frequencies.

For tyres, the damping is important, and the introduction of damping is presented in the Appendices.

3.1.2. *The projection theorem*

For two vector functions associated with two different natural frequencies $\omega^i \neq \omega^j$, a projection formula is found. Starting from the definition

$$\begin{aligned} \mathbf{b}\langle \mathbf{s}^i \rangle &= \mathrm{i}\omega^i\, \mathbf{m} \cdot \mathbf{s}^i, \\ \mathbf{b}\langle \mathbf{s}^j \rangle &= \mathrm{i}\omega^j\, \mathbf{m} \cdot \mathbf{s}^j, \end{aligned} \tag{37}$$

and using the Hermitian property, it is found that

$$\int_{-\pi}^{\pi} \mathbf{s}^{*i} \cdot \mathbf{b}\langle \mathbf{s}^j \rangle \,\mathrm{d}\theta = \mathrm{i}\omega^i \int_{-\pi}^{\pi} s^{*i} \cdot \mathbf{m} \cdot \mathbf{s}^j \,\mathrm{d}\theta,$$

$$\int_{-\pi}^{\pi} s^{*i} \cdot \mathbf{b}\langle \mathbf{s}^j \rangle \,\mathrm{d}\theta = \mathrm{i}\omega^j \int_{-\pi}^{\pi} \mathbf{s}^{*i} \cdot \mathbf{m} \cdot \mathbf{s}^j \,\mathrm{d}\theta. \tag{38}$$

The following property of projection holds:

$$\int_{-\pi}^{\pi} \mathbf{s}^{*i} \cdot \mathbf{m} \cdot \mathbf{s}^j \,\mathrm{d}\theta = 0. \tag{39}$$

Remark that this projection theorem is not the same with damping. Its expression in this case is derived in Appendix A.

3.1.3. *The solution construction*

Coming back to problem (23) with excitation sources $\mathbf{f}$,

$$\mathbf{f}\,(\omega, \theta) = (q_r(\omega, \theta), q_\theta(\omega, \theta), 0, 0)^{\mathrm{T}}. \tag{40}$$

Suppose that the natural frequencies ω^i are known, and the associated periodic functions $\mathbf{s}^i(\theta)$ are a basis of functions $([-\pi, \pi])^4$. The solution is decomposed in this basis:

$$\mathbf{s}(\omega, \theta) = \sum_{i=-\infty}^{\infty} a_i(\omega)\mathbf{s}^i(\theta). \tag{41}$$

Using property (27) of mode shapes, the equation to be solved becomes

$$\sum_{i=-\infty}^{\infty} \mathrm{i}\omega^i a_i(\omega)\mathbf{m}\cdot\mathbf{s}^i = \sum_{i=-\infty}^{\infty} \mathrm{i}\omega a_i(\omega)\mathbf{m}\cdot\mathbf{s}^i + \mathbf{f}(\omega). \tag{42}$$

This equation is solved with the multiplication by $\mathbf{s}^{*j}$ and integration on $[-\pi, \pi]$. Using property (39), and setting at first

$$e_j(\omega) = \frac{\int_{-\pi}^{\pi} \mathbf{s}^{*j}\cdot\mathbf{f}(\omega)\,\mathrm{d}\theta}{\int_{-\pi}^{\pi} \mathbf{s}^{*j}\cdot\mathbf{m}\cdot\mathbf{s}^j\,\mathrm{d}\theta}, \tag{43}$$

the solution is given by the scalar equation

$$\mathrm{i}(\omega^j - \omega)a_j = \mathbf{e}_j. \tag{44}$$

Finally, the solution of the vibration problem is given by

$$\begin{pmatrix} u_r(t,\theta) \\ u_\theta(t,\theta) \\ v_r(t,\theta) \\ v_\theta(t,\theta) \end{pmatrix} = \sum_{j=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e_j(\omega)\exp(\mathrm{i}\omega t)}{\mathrm{i}(\omega^j - \omega)}\,\mathrm{d}\omega\mathbf{s}^j(\theta). \tag{45}$$

3.2. PERTURBATION BY THE TREAD PATTERN

In previous publications, the density of the belt ρ had a uniform value. In this work, the belt is considered heterogeneous, and because of the rolling movement, ρ is a function of $\beta = \theta - \Omega t$, the position of the particle at $t = 0$.

The beam stiffness K and the bending stiffness D are also functions of β, although we consider that this heterogeneity is less important than the density heterogeneity.

Finally, the two stiffnesses w_r and w_θ are functions of β, and represent the elasticity of the rubber blocks. They are very small where the tread has a groove.

In equations (23), the mechanical characteristics are T-periodic. The period T is determined by the rolling speed and the tread pattern periodicity. If the tread pattern is m repetitions of a motif sequence, and if Ω is the angular speed of the wheel in rad/s, then

$$T = 2\pi/\Omega m, \tag{46}$$

where Ωm is also the fundamental pulsation of tread pattern impact on the road and is denoted by $\tilde{\Omega}$ in the following.

Equations (23) for a source term at pulsation ω are studied:

$$\mathbf{b}(t, \theta)\langle s\rangle - \mathbf{m}(t, \theta)\cdot\frac{\partial \mathbf{s}}{\partial t} = \mathbf{f}(\omega, \theta)\exp(\mathrm{i}\omega t). \tag{47}$$

A discrete Fourier transform in time is performed for $\mathbf{b}$ and $\mathbf{m}$:

$$\begin{aligned} \mathbf{b}(t, \theta) &= \sum_q \mathbf{b}_q(\theta)\exp(\mathrm{i}q\tilde{\Omega}t), \\ \mathbf{m}(t, \theta) &= \sum_q \mathbf{m}_q(\theta)\exp(\mathrm{i}q\tilde{\Omega}t). \end{aligned} \tag{48}$$

The equation is written again as

$$\sum_q \exp(\mathrm{i}q\tilde{\Omega}t)\,\mathbf{b}_q(\theta)\langle \mathbf{s}\rangle - \sum_q \exp(\mathrm{i}q\tilde{\Omega}t)\,\mathbf{m}_q(\theta)\cdot\frac{\partial \mathbf{s}}{\partial t} = \mathbf{f}(\omega, \theta)\exp(\mathrm{i}\omega t). \tag{49}$$

The vibration decomposition into mode shapes obtained in the previous section is used for solving equations (49) when $\mathbf{b}$ and $\mathbf{m}$ are disturbed by a small periodic oscillation caused by the tread heterogeneity.

3.2.1. *The order* 0

It is first assumed that the contribution of all the $\mathbf{b}_q$ and $\mathbf{m}_q$, $q \neq 0$ can be neglected in equation (49): only the time average of operators $\mathbf{b}_0$ and $\mathbf{m}_0$ are taken into account. Suppose that mode shapes and natural pulsations of the following problem have been computed with the theory of smooth tyre vibrations presented in the previous section:

$$\mathbf{b}_0\langle \mathbf{s}^i\rangle = \mathrm{i}\omega^i\mathbf{m}_0\cdot\mathbf{s}^i \tag{50}$$

These vectors are a basis and the solution can be decomposed in this basis:

$$\mathbf{s}(t, \theta) = \mathbf{s}_0(\theta)\exp(\mathrm{i}\omega t) = \sum_i a_i\mathbf{s}^i(\theta)\exp(\mathrm{i}\omega t). \tag{51}$$

This decomposition is introduced in equation (49), where only averaged contribution is modelled. Using the projection on the mode shapes, the solution at order 0 is simply

$$\mathbf{s}(t, \theta) = \mathbf{s}_0(\theta)\exp(\mathrm{i}\omega t) = \sum_i \frac{\mathrm{e}_i}{i(\omega^i - \omega)}\,\mathbf{s}^i(\theta)\exp(\mathrm{i}\omega t). \tag{52}$$

3.2.2. *Order* 1

Operators $\mathbf{b}(t, \theta)$ and $\mathbf{m}(t, \theta)$ are split into a time average and a perturbation considered of the first order (small comparative to the average):

$$\mathbf{b}(t, \theta) = \mathbf{b}_0(\theta) + \sum_{q \neq 0} \exp(\mathrm{i}q\tilde{\Omega}t)\,\mathbf{b}_q(\theta),$$

$$\mathbf{m}(t, \theta) = \mathbf{m}_0(\theta) + \sum_{q \neq 0} \exp(\mathrm{i}q\tilde{\Omega}t)\,\mathbf{m}_q(\theta). \tag{53}$$

The solution is also split into the preceding solution at order 0 and a correction of first order:

$$\mathbf{s}(t, \theta) = \mathbf{s}_0(\theta)\exp(\mathrm{i}\omega t) + \mathbf{s}_1(t, \theta). \tag{54}$$

Then the solution at order 1 is inserted again into equations:

$$\begin{aligned}&\left(\mathbf{b}_0 + \sum_{q \neq 0} \exp(\mathrm{i}q\tilde{\Omega}t)\,\mathbf{b}_q\right)\langle \mathbf{s}_0(\theta)\exp(\mathrm{i}\omega t) + \mathbf{s}_1(t, \theta)\rangle \\ &\quad -\left(\mathbf{m}_0 + \sum_{q \neq 0} \exp(\mathrm{i}q\tilde{\Omega}t)\,\mathbf{m}_q\right)\cdot\left(\mathrm{i}\omega\mathbf{s}_0\exp(\mathrm{i}\omega t) + \frac{\partial \mathbf{s}_1}{\partial t}\right) = \mathbf{f}(\omega, \theta)\exp(\mathrm{i}\omega t).\end{aligned} \tag{55}$$

Neglecting the terms of second order in equilibrium equations (product of two terms considered of first order), the correction is the solution of the problem

$$\mathbf{b}_0\langle \mathbf{s}_1\rangle - \mathbf{m}_0\cdot\frac{\partial \mathbf{s}_1}{\partial t} = \sum_{q \neq 0} \exp i\left(\omega t + q\tilde{\Omega}t\right)\left(i\omega m_q\cdot\mathbf{s}_0 - \mathbf{b}_q\langle \mathbf{s}_0\rangle\right). \tag{56}$$

The function $\mathbf{s}_1(t, \theta)$ is decomposed into frequency components:

$$\mathbf{s}_1(t, \theta) = \sum_{q \neq 0} \mathbf{s}_1(q, \theta)\exp\mathrm{i}(\omega t + q\tilde{\Omega}t). \tag{57}$$

Each frequency component verifies the equation

$$\mathbf{b}_0\langle \mathbf{s}_1(q, \theta)\rangle - \mathrm{i}(\omega + q\tilde{\Omega})\,\mathbf{m}_0\cdot\mathbf{s}_1(q, \theta) = \mathrm{i}\omega m_q\cdot\mathbf{s}_0 - \mathbf{b}_q\langle \mathbf{s}_0\rangle \tag{58}$$

The equation is solved by projection on the modal basis:

$$\mathbf{s}_1(q, \theta) = \sum_i b_i(q)\,\mathbf{s}^i(\theta). \tag{59}$$

Using this decomposition, it is finally found that

$$b_i(q) = \frac{1}{i(\omega^i - \omega - q\tilde{\Omega})} \frac{\displaystyle\int_{-\pi}^{\pi} (i\omega \mathbf{s}^{*i} \cdot \mathbf{m}_q \cdot \mathbf{s}_0 - \mathbf{s}^{*i} \cdot \mathbf{b}_q \langle \mathbf{s}_0 \rangle)\, \mathrm{d}\theta}{\displaystyle\int_{-\pi}^{\pi} \mathbf{s}^{*i} \cdot \mathbf{m} \cdot \mathbf{s}^i \, \mathrm{d}\theta}. \tag{60}$$

Now write the solution up to first order for a force at frequency ω:

$$\mathbf{s}(t,\theta) = \sum_i \exp(\mathrm{i}\omega t)\, a_i \mathbf{s}^i(\theta) + \sum_{q \neq 0} \exp(\mathrm{i}(\omega + q\tilde{\Omega})t) \sum_i b_i(q)\, \mathbf{s}^i(\theta). \tag{61}$$

The response contains component a_i at pulsation ω given by the theory of smooth tyres which is caused by external forces at frequency ω. The response also contains other components $b_i(q)$ at pulsations $(\omega + q\tilde{\Omega})$, $q \neq 0$, given by the perturbation theory. Terms $b_i(q)$ are caused by "self-excitation forces". They are a consequence of the disequilibrium of the approximation of order 0 when heterogeneity is taken into account. The effect of the rotation of the groove is to give forces whose pulsation is the one of the order 0 translated by $q\tilde{\Omega}$. This means that the modal analysis, that assumes that the response contains only the frequency ω, is not applicable to analyze a rolling tyre.

Rewriting the solution

$$\mathbf{s}(t,\theta) = \exp(\mathrm{i}\omega t) \sum_i \left(a_i + \sum_{q=0} \exp(iq\tilde{\Omega}t)\, b_i(q) \right) \mathbf{s}^i(\theta), \tag{62}$$

it is shown that it is the product of an oscillating function by a function of t and θ, T-periodic in t and 2π-periodic in θ. This decomposition will be used again later.

3.3. ROLLING NOISE SPECTRUM

Now consider that tread heterogeneity also contributes partially to a flat spectrum of tyre noise, which will be explained with the physical meaning of equation (61).

Suppose that the natural frequencies of smooth tyres were calculated for rolling and non-rolling tyres. Now try to apply qualitatively these results to commercial tyres. In the preceding publications, damped resonances are measured for non-rolling commercial tyres, at least up to 400 Hz. At high frequencies (above 800 Hz), these resonances are not measurable; it is usually explained by a high damping of vibrations caused by the viscosity of rubber. The theory of smooth tyres seems to give a good agreement in the case of non-rolling tyres. But the theory of smooth tyre does not apply for rolling tyres; damped resonance was not visible on rolling noise spectra of tyres with a tread pattern even at low frequency. For tyres with a randomized tread pattern, the noise spectrum appears flat.

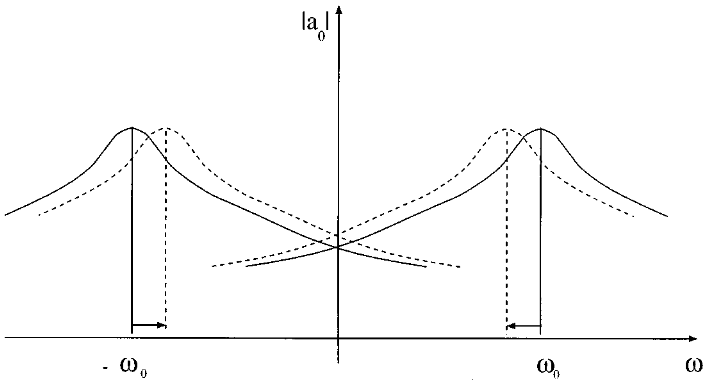

Figure 2. Change of natural frequencies caused by rolling speed.

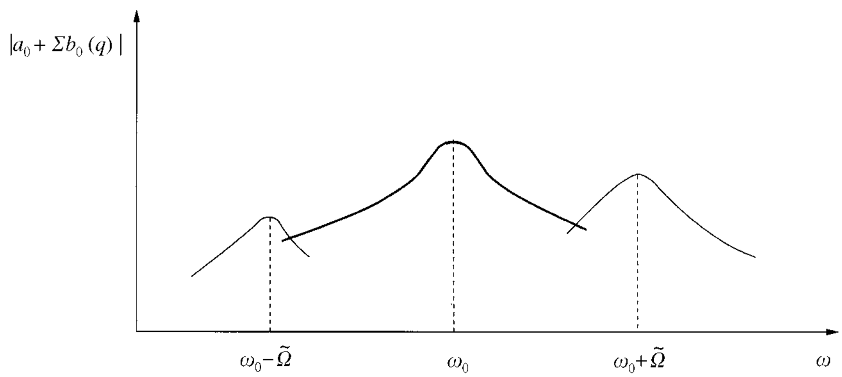

Figure 3. The tread pattern effect (periodic tread pattern).

Now examine the physical meaning of equation (61). As a first approximation, the tyre can be considered to be smooth: amplifications occurs if the excitation frequency ω is near a natural pulsation of the smooth tyre ω^i. In this case, the solution at order 0, $\exp(i\omega^i t)\mathbf{s}_0(\omega^i, \theta)$, is of greater amplitude. Then, the perturbation induced by the heterogeneity is proportional to the order 0 solution. This perturbation reproduces the amplifications at frequencies ω^i at the other pulsations $(\omega^i + q\tilde{\Omega})$.

Now illustrate this with the simplified case where the solution at order 0 contains only one excited mode, a_0. Figure 2 shows the modal amplitude of mode 0 obtained with the smooth tyre theory. The plain and dashed lines show result for two different rolling speeds. As mentioned before, ω^0, the natural frequency of this mode changes when the rolling speed increases. In Figures 3 and 4, the tread pattern is taken into account. Modal amplitude of the first mode shows a centre peak that correspond to natural frequency ω^0, and other peaks that correspond to frequencies $(\omega^0 + q\tilde{\Omega})$. If the fundamental frequency of the tread pattern impact $\tilde{\Omega}$ is high (for example for tyres with a periodic pattern, and rolled at high speed), the

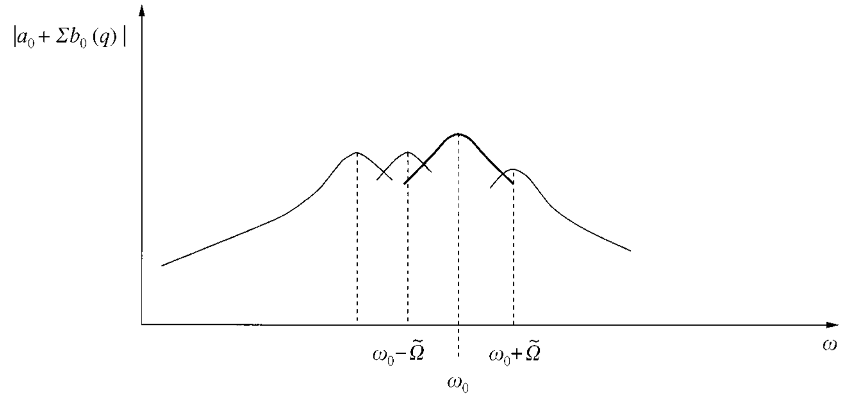

Figure 4. The tread pattern effect (randomized tread pattern).

frequencies $(\omega^0 + q\tilde{\Omega})$ are well separated, and the contributions of the terms $b_0(q)$ do not overlap the contribution a_0 (see Figure 3). On the contrary, if the fundamental frequencies is small (for example for tyres with randomized tread pattern), the frequencies $(\omega^0 + q\tilde{\Omega})$ are close to each other and overlap (see Figure 4). The spectrum of rolling noise appears flat.

4. THE THEORETICAL SOLUTION

The perturbation theory presented above is approximate. In the following, the extension of these results to the general case is investigated. The effect of tread pattern on tyre vibrations is developed for the heterogeneous "circular ring model". This will present the global theoretical frame of the study and will lead to a proposal for the experimental characterization of the tyre rolling noise.

The effective numerical analysis is then developed in Section 5.

4.1. THE METHOD

In the previous section, it was seen that if the tyre is modelled by a time periodic system, then all the frequencies $\omega_q = (\varphi + q\tilde{\Omega})$, q the relative integer, are mixed by the system.

For a fixed value of φ, the following excitation contains all the frequencies ω_q:

$$\sum_{q=-\infty}^{\infty} \hat{q}_r(\omega_q, \theta)\exp(\mathrm{i}\omega_q t) = \exp(\mathrm{i}\varphi t) \sum_{q=-\infty}^{\infty} \hat{q}_r(\varphi + q\tilde{\Omega}, \theta)\exp(\mathrm{i}q\tilde{\Omega}t)$$

$$= \exp(\mathrm{i}\varphi t)\, q_r(\varphi, t, \theta), \qquad (63)$$

where $q_r(\varphi, t, \theta)$ is T-periodic in t for all values of φ. The same decomposition is done for $q_\theta(t, \theta)$.

It is possible to look for the solution $u_r(t, \theta) = \exp(\mathrm{i}\varphi t)u_r(\varphi, t, \theta)$ with $u_r(\varphi, t, \theta)$ T-periodic (and the same form for u_θ, v_r, v_θ), that is, the same set of frequencies as the excitation.

4.1.1. *The diagonalization problem*

Introducing these forms to the equilibrium equations (23), simplifing by $\exp(\mathrm{i}\varphi t)$ and rearranging, yields the following system where all functions are periodic in t and in θ:

$$\underbrace{-\frac{1}{a^2}(D\chi_\theta)'' + \frac{1}{a}K\varepsilon_\theta + \frac{1}{a}t_i\gamma_z' + (k_r + w_r(\beta))u_r}_{\text{stress}} + \underbrace{\rho(\beta)(\dot{v}_r + \Omega v_r' - \Omega v_\theta)}_{\text{gyroscopic}}$$

$$= -\mathrm{i}\varphi\,\rho(\beta)v_r + q_r, \tag{64}$$

$$\underbrace{-\frac{1}{a^2}(D\chi_\theta)' - \frac{1}{a}(K\varepsilon_\theta)' + \frac{1}{a}t_i\gamma_z + (k_\theta + w_\theta(\beta))u_\theta}_{\text{stress}} + \underbrace{\rho(\beta)(\dot{v}_\theta + \Omega v_\theta' + \Omega v_r)}_{\text{gyroscopic}}$$

$$= -\mathrm{i}\varphi\;\rho(\beta)v_\theta + q_\theta, \tag{65}$$

$$\underbrace{-\rho(\beta)(\dot{u}_r + \Omega u_r' - \Omega u_\theta)}_{\text{gyroscopic}} + \underbrace{\rho(\beta)v_r}_{\text{impulsion}} = \mathrm{i}\varphi\;\rho(\beta)u_r \tag{66}$$

$$\underbrace{-\rho(\beta)(\dot{u}_\theta + \Omega u_\theta' + \Omega u_r)}_{\text{gyroscopic}} + \underbrace{\rho(\beta)v_\theta}_{\text{impulsion}} = \mathrm{i}\varphi\rho(\beta)u_\theta. \tag{67}$$

The problem to be solved has the following structure:

$$\mathbf{b}\langle\mathbf{s}\rangle = \mathrm{i}\varphi\;\mathbf{m}\cdot\mathbf{s} + \mathbf{f} \tag{68}$$

where $\mathbf{b}(\beta)$ is a (4×4) matrix of linear differential operators both in time and polar angle with time periodic coefficients, and $\mathbf{m}(\beta)$ is a matrix with time periodic components. $\mathbf{f}$ is the source term of the equations

$$\mathbf{f}(\varphi, t, \theta) = (q_r, q_\theta, 0, 0)^{\mathrm{T}}. \tag{69}$$

The associated diagonalization problem for the function of $([0, T]\times[-\pi, \pi])^4$, periodic in θ and periodic in t is found to be

$$\mathbf{b}\langle\mathbf{s}\rangle = \mathrm{i}\varphi\;\mathbf{m}\cdot\mathbf{s}. \tag{70}$$

Four functions $\delta u_r, \delta u_\theta, \delta v_r, \delta v_\theta$ of $[0, T] \times [-\pi, \pi]$, periodic in θ and periodic in t are chosen, and the four component vector is denoted by $\delta\mathbf{s}$. Multiplying each equation by the conjugate of a component of $\delta\mathbf{s}$, integrating on $[0, T] \times [-\pi, \pi]$, and adding all the equations, produces the weak formulation:

$$\int_0^T \int_{-\pi}^{\pi} \delta\mathbf{s}^* \cdot \mathbf{b}\langle\mathbf{s}\rangle \, d\theta \, dt = i\varphi \int_0^T \int_{-\pi}^{\pi} \delta\mathbf{s}^* \cdot \mathbf{m} \cdot \mathbf{s} d\theta \, dt. \tag{71}$$

It can be shown that both the form $\mathbf{b}\langle.\rangle$ and the matrix i$\mathbf{m}$ are Hermitian.

The stress terms of equations (64, 65) lead to the integral of elastic form, also a symmetric term. With the same notations as in the previous section for U, it is found that

$$\begin{aligned} &\int_0^T \int_{-\pi}^{\pi} \left[-\frac{1}{a^2}(D\chi_\theta)'' + \frac{1}{a} K\varepsilon_\theta + \frac{1}{a} t_i \gamma_z' + (k_r + w_r(\beta)) u_r \right] \overline{\delta u_r} \, d\theta \, dt \\ &+ \int_{-\pi}^{\pi} \left[-\frac{1}{a^2}(D\chi_\theta)' - \frac{1}{a}(K\varepsilon_\theta)' + \frac{1}{a} t_i \gamma_z + (k_\theta + w_\theta(\beta)) u_\theta \right] \overline{\delta u_\theta} \, d\theta \, dt \\ &= \int_0^T U(\delta\mathbf{s}, \mathbf{s}) \, dt. \end{aligned} \tag{72}$$

The impulsion terms of equations (66, 67) lead to the integral of kinetic form, also a symmetric term,

$$\int_0^T \int_{-\pi}^{\pi} (\overline{\delta v_r} \, \rho v_r + \overline{\delta v_\theta} \rho v_\theta) \, d\theta \, dt. \tag{73}$$

The gyroscopic terms in equation (66, 65) are integrated by parts, either in angle, or in time. Using the periodicity properties of all the functions, and the property of the linear mass $\dot{\rho} + \Omega\rho' = 0$, it is firstly found that

$$\begin{aligned} &-\int_0^T \int_{-\pi}^{\pi} \rho \, [\overline{\delta v_r}(\dot{u}_r + \Omega u_r' - \Omega u_\theta) + \overline{\delta v_\theta}(\dot{u}_\theta + \Omega u_\theta' + \Omega u_r)] \, d\theta \, dt \\ &= \int_0^T \int_{-\pi}^{\pi} \rho \, [u_r \overline{(\delta \dot{v}_r + \Omega \delta v_r' - \Omega \delta v_\theta)} + u_\theta \overline{(\delta \dot{v}_\theta + \Omega \delta v_\theta' + \Omega \delta v_r)}] \, d\theta \, dt. \end{aligned} \tag{74}$$

Then, the gyroscopic terms of equations (64, 65), are added, and a symmetric form is also obtained. With the same notations as in the last section for G, it is found that:

$$\int_0^T \int_{-\pi}^{\pi} \rho [\overline{\delta u_r}(\dot{v}_r + \Omega v_r' - \Omega v_\theta) + \overline{\delta u_\theta}(\dot{v}_\theta + \Omega v_\theta' + \Omega v_r)] \, d\theta \, dt$$

$$- \int_0^T \int_{-\pi}^{\pi} \rho [\overline{\delta v_r}(\dot{u}_r + \Omega u_r' - \Omega u_\theta) + \overline{\delta v_\theta}(\dot{u}_\theta + \Omega u_\theta' + \Omega u_r)] \, d\theta \, dt$$

$$= \int_0^T G(\delta \mathbf{s}, \mathbf{s}) \, dt + \int_0^T \int_{-\pi}^{\pi} \rho [\overline{\delta u_r} \dot{v}_r + \overline{\delta u_\theta} \dot{v}_\theta + u_r \dot{\overline{\delta v_r}} + u_\theta \dot{\overline{\delta v_\theta}}] \, d\theta \, dt \quad (75)$$

Finally, equation (76) shows that the operator $\mathbf{b}\langle . \rangle$ is Hermitian:

$$\int_0^T \int_{-\pi}^{\pi} \delta \mathbf{s}^* \cdot \mathbf{b}\langle \mathbf{s} \rangle \, d\theta \, dt = \int_0^T U(\delta \mathbf{s}, \mathbf{s}) \, dt + \int_0^T \int_{-\pi}^{\pi} (\overline{\delta v_r} \rho v_r + \overline{\delta v_\theta} \rho v_\theta) \, d\theta \, dt$$

$$+ \int_0^T G(\delta \mathbf{s}, \mathbf{s}) \, dt + \int_0^T \int_{-\pi}^{\pi} \rho [\overline{\delta u_r} \dot{v}_r + \overline{\delta u_\theta} \dot{v}_\theta + u_r \dot{\overline{\delta v_r}} + u_\theta \dot{\overline{\delta v_\theta}}] \, d\theta \, dt. \quad (76)$$

Matrix $\mathbf{m}$ is antisymmetric and therefore i$\mathbf{m}$ is also Hermitian.

As a consequence, for two sets of functions associated with two different values $\varphi^i \neq \varphi^j$, it is found that the following property holds:

$$\int_0^T \int_{-\pi}^{\pi} \mathbf{s}^{*j} \cdot \mathbf{m} \cdot \mathbf{s}^i \, d\theta \, dt = 0. \quad (77)$$

4.1.2. *The solution construction*

Returning to problem (68), suppose that φ^i are known, and associated periodic functions $\mathbf{s}^i$ are a basis of the functions $([0, T] \times [-\pi, \pi])^4$. The solution is decomposed on this basis:

$$\mathbf{s}(\varphi, t, \theta) = \sum_{i=-\infty}^{\infty} a_i(\varphi) \mathbf{s}^i(t, \theta). \quad (78)$$

Using the property of functions $\mathbf{s}^i$, the equation to be solved becomes

$$\sum_{i=-\infty}^{\infty} \mathrm{i} \varphi^i a_i \mathbf{m} \cdot \mathbf{s}^i = \sum_{i=-\infty}^{\infty} \mathrm{i} \varphi a_i \mathbf{m} \cdot \mathbf{s}^i + \mathbf{f}. \quad (79)$$

This equation is solved with the multiplication by $\mathbf{s}^{*j}$ and integration on $[0, T] \times [-\pi, \pi]$. Using the property (77), and setting firstly:

$$e_j = \frac{\int_0^T \int_{-\pi}^{\pi} \mathbf{s}^{*j} \cdot \mathbf{f} \, d\theta \, dt}{\int_0^T \int_{-\pi}^{\pi} \mathbf{s}^{*j} \cdot \mathbf{m} \cdot \mathbf{s}^j \, d\theta \, dt}, \quad (80)$$

the solution is given by the reduced equation

$$\mathrm{i}(\varphi^j - \varphi) a_j = e_j. \quad (81)$$

Then, the solution of the diagonalization problem with a source term is

$$\mathbf{s}(\varphi, t, \theta) = \sum_{i=-\infty}^{\infty} \frac{e_i}{\mathrm{i}(\varphi^i - \varphi)} \mathbf{s}^i(t, \theta). \tag{82}$$

Finally, the solution of the vibration problem is given by

$$\begin{pmatrix} u_r(t, \theta) \\ u_\theta(t, \theta) \\ v_r(t, \theta) \\ v_\theta(t, \theta) \end{pmatrix} = \int_{-\tilde{\Omega}/2}^{\tilde{\Omega}/2} \sum_{i=-\infty}^{\infty} \frac{e_i \exp(\mathrm{i}\varphi t)}{\mathrm{i}(\varphi^i - \varphi)} \mathbf{s}^i(t, \theta)\, \mathrm{d}\varphi. \tag{83}$$

This theoretical formula gives the vibration decomposition. The values of φ^i are real numbers when the dissipation is not taken into account, but for real tyres, φ^i should have an imaginary part that models the damping.

4.2. EXPERIMENTAL ANALYSIS OF THE ROLLING NOISE SPECTRUM

The above analysis suggests the use of a representation different from the spectrum for the experimental study of the sound radiated by a rolling tyre. Suppose that the signal $p(t)$ is measured at one point with a microphone. Because all the frequencies $(\varphi + q\tilde{\Omega})$ are mixed by the system, the following function $p(\varphi, t)$ should be represented. For any time t in $[0, T]$ and any phase φ in $[-\tilde{\Omega}/2, \tilde{\Omega}/2]$, its value is given by

$$\begin{aligned} p(t, \varphi) &= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} p(t + nT) \exp(-\mathrm{i}\varphi nT) \\ &= \exp(\mathrm{i}\varphi t) \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} p(t + nT) \exp(-\mathrm{i}\varphi(t + nT)). \end{aligned} \tag{84}$$

Note that the component $p(t, 0)$ is the periodic component of the noise of a rolling tyre, and has already been widely used by Walker [16] in rolling noise analysis.

By linearity, it is the solution of equations (23) with excitation forces:

$$\begin{aligned} \mathbf{q}(t, \varphi, \theta) &= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \mathbf{q}(t + nT, \theta) \exp(-\mathrm{i}\varphi nT) \\ &= \exp(\mathrm{i}\varphi t) \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \mathbf{q}(t + nT, \theta) \exp(-\mathrm{i}\varphi(t + nT)). \end{aligned} \tag{85}$$

The function $p(t, \varphi)$ is the product of $\exp(\mathrm{i}\varphi t)$ by a periodic function of time. Its value is then calculable with the Bloch wave theory. Suppose then that after signal processing, the value of $p(t, \varphi)$ is known. From this value, it will be shown that it is possible to measure the numbers φ^i of a rolling tyre.

Like the frequency representation of the spectrum that allows for the identification of natural frequencies in experiments, the Bloch wave presentation of experimental or numerical data makes the dynamics of a rolling tyre easy to understand. This representation is a 3-D diagram of the acoustic pressure p as a function of time and phase angle. In the decomposition (83), it is shown that an amplification of the vibrations occurs if φ is near a value φ^i; phase angles φ^i are observed by peaks on the φ-axis.

Note that when the tyre is rolling on a flat surface, it is seen that the excitation source is the tread pattern excitation, having a discrete spectrum containing all pulsations $q\tilde{\Omega}$. It means that only $\mathbf{q}(t, 0, \theta)$ is non-zero. This excitation source is then amplified if many values of φ^i near 0 exist (and if the associated vibrations are not very damped, or if they can radiate sound).

5. THE NUMERICAL COMPUTATION

In the case of a mechanical mode discretized into a great number of degrees of freedom, the calculation of the numbers φ^i and the periodic functions with Bloch theory can be of very high computational cost, especially if the period T is long (randomized tread pattern). For numerical simulation, it is preferable to use the Floquet theory.

5.1. THE DIAGONALIZATION PROBLEM

Now examine the relation between Bloch wave theory and Floquet theory. Remember that by definition, with a phase φ^i and a Bloch mode shapes $\mathbf{s}^i(t, \theta) = (u^i_r, u^i_\theta, v^i_r, v^i_\theta)^T$ T-periodic in time, it is possible to construct a solution of free propagation in the system:

$$\mathbf{l}^i(t, \theta) = \exp(\mathrm{i}\varphi^i t)\,\mathbf{s}^i(t, \theta). \tag{86}$$

This solution satisfies the following important properties:

$$\mathbf{l}^i(T, \theta) = \exp(\mathrm{i}\varphi^i T)\,\mathbf{l}^i(0, \theta),$$

$$\mathbf{l}^i(0, \theta) = \mathbf{s}^i(0, \theta). \tag{87}$$

Denote by $P[\ ]$, the linear operator that integrates the propagation equations without excitation source. Knowing an initial value of Cauchy type at time $t = 0$, $\mathbf{l}(0, \theta)$, $P[\ ]$ gives the solution at time T as a linear form of the initial condition:

$$\mathbf{l}(T, \theta) = P[\mathbf{l}(0, \theta)]. \tag{88}$$

The above analysis suggests one to calculate and diagonalize P: a method for the calculation and the diagonalization of P is proposed in Appendix C. Suppose that functions $l_0^i(\theta)$ and Floquet coefficients λ^i are known:

$$P[\mathbf{l}_0^i(\theta)] = \lambda^i \mathbf{l}_0^i(\theta). \tag{89}$$

The value at time 0 of the Bloch shape functions verifies the eigenvalue problem (89): $\mathbf{l}_0^i(\theta) = \mathbf{s}^i(0, \theta)$ is the solution of the diagonalization problem (89) with eigenvalue $\lambda^i = \exp(\mathrm{i}\varphi^i T)$.

Reciprocally, it is possible to define the solution of the free propagation problem with initial value $\mathbf{l}_0^i(\theta)$: $\mathbf{l}^i(0, \theta) = \mathbf{l}_0^i(\theta)$. Setting $\mathbf{s}^i(t, \theta) = \exp(-\mathrm{i}\varphi^i t)\mathbf{l}^i(t, \theta)$, function $\mathbf{s}^i(t, \theta)$ is T-periodic, and is the solution of the diagonalization problem (70).

It has been shown that both sets of functions $\mathbf{s}^i(t, \theta)$ and $\mathbf{l}_0^i(\theta)$ are equivalent, and the following relationship holds:

$$\mathbf{l}_0^i(\theta) = \mathbf{s}^i(0, \theta). \tag{90}$$

Numbers $\lambda^i = \exp(\mathrm{i}\varphi^i T)$ are called the Floquet coefficients. The knowledge of both these numbers, and the values of the Floquet shapes $\mathbf{l}_0^i(\theta)$ for a given t is equivalent to the Bloch wave analysis. The phase of λ^i is the change of phase of the solution of free propagation in one period T, its modulus being the amplification. As we have dissipative problem, the modulus is lower than one.

5.2. THE STATIONARY CONDITION

When a time integration algorithm is used, a Cauchy type initial condition is needed. In the case of stationary rolling, this initial condition is not known. It will be replaced by a relation between initial state and state at time T. Now derive this relation called stationary condition.

Suppose an excitation at pulsation $\omega = (\varphi + q_0\tilde{\Omega})$, φ and q_0 fixed, and compare the state vector field in the tyre between two instants separated by one period T. It has been shown in section 4 that the response contains all the frequencies $\omega_q = (\varphi + q\tilde{\Omega})$, q relative integer. Using the frequency decomposition of the solution for this excitation,

$$\mathbf{s}(t, \theta) = \sum_{q=-\infty}^{\infty} \mathbf{s}_q(\theta) \exp(i(\varphi + q\tilde{\Omega})t), \tag{91}$$

it is found that

$$\mathbf{s}(t + T, \theta) = \sum_{q=-\infty}^{\infty} \mathbf{s}_q(\theta) \exp(i(\varphi + q\tilde{\Omega})\,(t + T)),$$

$$\mathbf{s}(t + T, \theta) = \exp(\mathrm{i}\varphi T) \sum_{q=-\infty}^{\infty} \mathbf{s}_q(\theta) \exp(\mathrm{i}(\varphi + q\tilde{\Omega})t), \tag{92}$$

$$\mathbf{s}(t + T, \theta) = \exp(\mathrm{i}\varphi T)\,\mathbf{s}(t, \theta).$$

This condition is the definition for stationary state of vibrations induced by excitation source at pulsation ω. This condition is the same for any value of q_0 in the expression of the frequency of excitation sources, $(\varphi + q_0\tilde{\Omega})$.

5.3. THE DECOMPOSITION FORMULA

The equilibrium equations (23) are linear equations with a source term. The solution is the sum of a solution of the equations with source term and a special initial condition, and a combination of solutions of the equations without source term. For the special initial condition, a zero displacement and a zero relative speed are chosen. With this condition and a time integration algorithm, it is possible to get the solution at time T, $z(T, \theta)$, of the equation system (23), with sources q_r and q_θ oscillating at pulsation in the set $\omega_q = (\varphi + q\tilde{\Omega})$, φ fixed and q relative integer.

The stationary condition must be satisfied. Of course, the solution z has no reason to verify this condition:

$$\mathbf{z}(T, \theta) \neq \exp(\mathrm{i}\varphi T)\mathbf{z}(0, \theta). \tag{93}$$

For this reason, a combination of solutions of the equations without source term is added to this particular solution. This combination is chosen as a combination of the functions $\mathbf{l}^i(t, \theta)$, that verify the initial condition $\mathbf{l}^i(0, \theta) = \mathbf{l}_0^i(\theta)$, where $\mathbf{l}_0^i(\theta)$ is a solution of the diagonalization problem (89):

$$\mathbf{s}(t, \theta) = \mathbf{z}(t, \theta) + \sum_{i=1}^{I} a_i(\omega)\mathbf{l}^i(t, \theta). \tag{94}$$

The numbers a_i must be calculated to have the stationarity condition. The stationarity condition is then derived for this expression:

$$\mathbf{s}(0, \theta) = \mathbf{z}(0, \theta) + \sum_{i=1}^{I} a_i\mathbf{l}^i(0, \theta) = \sum_{i=1}^{I} a_i\mathbf{l}_0^i(\theta),$$

$$\mathbf{s}(T, \theta) = \mathbf{z}(T, \theta) + \sum_{i=1}^{I} a_i\mathbf{l}^i(T, \theta) = \mathbf{z}(T, \theta) + \sum_{i=1}^{I} a_i\lambda^i\mathbf{l}_0^i(\theta), \tag{95}$$

$$\sum_{i=1}^{I} a_i(\exp(\mathrm{i}\varphi T) - \lambda^i)\mathbf{l}_0^i(\theta) = \mathbf{z}(T, \theta).$$

In order to solve this system, a projection formula is developed in Appendix B. The result is that for $i \neq j$,

$$\int_{-\pi}^{\pi} \mathbf{l}_0^{j\mathrm{T}}(\theta)\cdot\mathbf{m}(T, \theta)\cdot\mathbf{l}_0^i(\theta)\,\mathrm{d}\theta = 0. \tag{96}$$

Using this result, the expression of a_i is found:

$$a_i = \frac{1}{(\exp(\mathrm{i}\varphi T) - \lambda^i)} \frac{\displaystyle\int_{-\pi}^{\pi} \mathbf{l}_0^{i\mathrm{T}}(\theta)\cdot\mathbf{m}(T,\theta)\cdot\mathbf{z}(T,\theta)\,\mathrm{d}\theta}{\displaystyle\int_{-\pi}^{\pi} \mathbf{l}_0^{i\mathrm{T}}(\theta)\cdot\mathbf{m}(T,\theta)\cdot\mathbf{l}_0^{i}(\theta)\,\mathrm{d}\theta}. \tag{97}$$

The solution of this system gives the values of a_i, and as a consequence, the value at time $t = 0$ of the solution of equations (23):

$$\mathbf{s}(t,0) = \sum_i a_i \mathbf{l}_0^i(\theta). \tag{98}$$

The complete solution at other times $t \in]0, T[$ is obtained by time integration of this initial condition.

5.4. RETURN TO BLOCH WAVE ANALYSIS

Finally, it is also possible to find the value of φ^i and $e_i(\varphi)$ defined in the Bloch wave theory from λ^i and $a_i(\omega)$ calculated using the Floquet theory.

Firstly, the equation $\exp(\mathrm{i}\varphi^i T) = \lambda^i$ has to be solved. It has only one solution φ^i with an argument in $] -\tilde{\Omega}/2, \tilde{\Omega}/2]$.

Then, for a given φ, it has been shown that the excitation spectrum contains all the frequencies $\omega_q = (\varphi + q\tilde{\Omega})$. In order to derive $e_i(\varphi)$, all the values $a_i(\omega_q)$ are necessary. By linearity, they can be calculated at once, if the solution $\mathbf{z}_q(t,\theta)$ is the response of the structure to the excitation forces $\sum_q q_r(\omega_q, \theta)$ and $\sum_q q_\theta(\omega_q, \theta)$. Relation (97) can be used:

$$a_i(\varphi) = \frac{1}{(\exp(\mathrm{i}\varphi T) - \lambda^i)} \frac{\displaystyle\int_{-\pi}^{\pi} \mathbf{l}_0^{i\mathrm{T}}(\theta)\cdot\mathbf{m}(T,\theta)\cdot\sum_q \mathbf{z}_q(T,\theta)\,\mathrm{d}\theta}{\displaystyle\int_{-\pi}^{\pi} \mathbf{l}_0^{i\mathrm{T}}(\theta)\cdot\mathbf{m}(T,\theta)\cdot\mathbf{l}_0^{i}(\theta)\,\mathrm{d}\theta}. \tag{99}$$

Comparing the expression at $t = 0$ of the solution given using the Floquet theory and the Bloch theory, it is found that

$$\mathbf{s}(t,0) = \sum_i a_i \mathbf{l}_0^i(\theta) = \sum_i \frac{e_i}{\mathrm{i}(\varphi^i - \varphi)} \mathbf{s}^i(0,\theta). \tag{100}$$

Remembering that $\mathbf{l}_0^i(\theta) = \mathbf{s}^i(0,\theta)$, the value of e_i is shown to be

$$e_i = a_i(\mathrm{i}\varphi^i - i\varphi). \tag{101}$$

In consequence, it is possible to compare numerical simulations obtained by the Floquet theory with measurements obtained by the signal processing derived from the Bloch theory of section 4.2.

6. CONCLUSION

After examination of the theories of vibrations in linear systems with time periodic coefficients, three main results were obtained for rolling tyres:

1. The effect of the tread pattern on rolling tyre vibrations was investigated. It was shown that for randomized tyres, this effect makes the noise spectrum appear flat, as measured in experiments (see Figure 4).
2. From an experimental point of view, the analysis of rolling tyre noise could be easier to understand if the results of the spectrum are presented in a suitable way to take into account the impact periodicity of the rubber blocks. It was suggested in section 4 to plot the following pressure signal for the values of φ between 0 and the half of the fundamental pulsation of the rubber block impact $\tilde{\Omega}/2$ and for the value of t between 0 and the period of rubber block impact T:

$$p(\varphi, t) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} p(t + nT) \exp(-i\varphi nT). \qquad (102)$$

   This extends a result already used in reference [16] for tyre noise analysis.
3. From a computational point of view, calculation of vibrations of a smooth tyre is not sufficient for the dynamical analysis of a rolling tyre with a tread pattern.

   - A self-excitation caused by the tread pattern has to be included for the study of nearly smooth tyres (in section 3 and Appendix A).
   - Modal analysis should be replaced by the Floquet analysis for tyre with a winter type pattern. This analysis is explained in the case of the "circular ring model" (in section 5 and in Appendices B and C).

It is considered that the extension to real structures of these methods is possible with the finite element method.

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APPENDIX A: COMPUTATION OF NATURAL FREQUENCIES OF A SMOOTH TYRE

A.1. THE DISCRETIZATION OF THE PROBLEM

The problem to be solved is the following one (see section 3.1.1):

$$\begin{cases} -\dfrac{1}{a^2}(D\chi_\theta)'' + \dfrac{1}{a}K\varepsilon_\theta + \dfrac{1}{a}t_i\gamma_z' + (k_r + w_r)u_r & + \Omega\rho(v_r' - v_\theta) & = \mathrm{i}\omega(-\rho)v_r \\ -\dfrac{1}{a^2}(D\chi_\theta)' - \dfrac{1}{a}(K\varepsilon_\theta)' + \dfrac{1}{a}t_i\gamma_z + (k_\theta + w_\theta)u_\theta & + \Omega\rho(v_\theta' + v_r) & = \mathrm{i}\omega(-\rho)v_\theta \\ -\Omega\rho(u_r' - u_\theta) & + \rho v_r & = \mathrm{i}\omega\rho u_r \\ -\Omega\rho(u_\theta' + u_r) & + \rho v_\theta & = \mathrm{i}\omega\rho u_\theta \end{cases} \tag{A1}$$

In this section, a discrete formulation of the above problem is given. The equations verified by the discretized field vectors $\mathbf{U}(\omega)$ and $\mathbf{V}(\omega)$ that represent the values of the displacement field $\mathbf{u}(\omega, \theta)$ and $\mathbf{v}(\omega, \theta)$ in fixed point $\theta = \theta^i$ are derived:

$$\mathbf{U}(\omega) = \begin{pmatrix} u_r & (\omega, \theta^1) \\ u_\theta & \\ \cdots & \\ u_r & (\omega, \theta^i) \\ u_\theta & \\ \cdots & \end{pmatrix}, \qquad \mathbf{V}(\omega) = \begin{pmatrix} v_r & (\omega, \theta^1) \\ v_\theta & \\ \cdots & \\ v_r & (\omega, \theta^i) \\ v_\theta & \\ \cdots & \end{pmatrix}. \tag{A2}$$

Each component $\alpha(= r, \theta)$ of the displacement field $\mathbf{u}(\omega, \theta)$ is approximated by the nodal shape function $N_\alpha^i(\theta)$ by the formula

$$u_\alpha(\omega, \theta) = \sum_i N_\alpha^i(\theta) u_\alpha(\omega, \theta^i). \tag{A3}$$

The nodal shape functions $N_\alpha^i(\theta)$ are continuous. They follow three conditions: (1) $N_\alpha^i(\theta^i) = 1$, (2) $N_\alpha^i(\theta^j) = 0$ if $i \neq j$, and (3) the consistency condition.

A.1.1. Rigidity matrix

With the weak formulation of the problem, the elastic energy form has been defined:

$$U(\delta \mathbf{s}, \mathbf{s}) = \int_{-\pi}^{\pi} (\overline{\delta\chi_\theta} D\chi_\theta + \overline{\delta\varepsilon_\theta} K\varepsilon_\theta + \overline{\delta\gamma_z} t_i \gamma_z + \overline{\delta u_r}(k_r + w_r) u_r + \overline{\delta u_\theta}(k_\theta + w_\theta) u_\theta)\, \mathrm{d}\theta. \tag{A4}$$

Its discretized formulation is the rigidity matrix $\mathbf{K}$.

A.1.2. Gyroscopic matrix

The gyroscopic form has also been introduced:

$$G(\delta \mathbf{s}, \mathbf{s}) = \Omega \int_{-\pi}^{\pi} \rho\, [\overline{\delta u_r} \cdot (v_r' - v_\theta) + \overline{\delta u_\theta} \cdot (v_\theta' + v_r)]\, \mathrm{d}\theta$$

$$+ \Omega \int_{-\pi}^{\pi} \rho\, [u_r(\overline{\delta v_r'} - \overline{\delta v_\theta}) + u_\theta(\overline{\delta v_\theta' + \delta v_r})]\, \mathrm{d}\theta. \tag{A5}$$

The discretized formulation of the first integral is called $\Omega\mathbf{G}$, and the discretized formulation of the second integral is simply $\Omega\mathbf{G}^{\mathrm{T}}$.

A.1.3. *Mass matrix*

The mass matrix **M** is defined as usual.

A.1.4. *Damping matrix*

If it is assumed that relaxation times are very short in comparison with a characteristic time of the problem, then the Rayleigh model for damping can be used:

$$\boldsymbol{\sigma} = (1-\varepsilon_\theta)\sigma_{th}\begin{pmatrix} -\gamma_z \\ (1+\varepsilon_\theta)\end{pmatrix} = t_i(\mathbf{e}_\theta - \gamma_z\mathbf{e}_r) + \left(K\varepsilon_\theta + v\left.\frac{\partial\varepsilon_\theta}{\partial t}\right|_\beta\right)\mathbf{e}_\theta. \tag{A6}$$

This formulation has the big advantage of being "instantaneous" and then only the actual position and speed are needed to predict the next time step. This behaviour must be extended to the bending D, the tread stiffnesses w_r and w_θ, and the sidewall stiffnesses k_r and k_θ. Damping is responsible for the new term:

$$\begin{aligned} C(\delta\mathbf{s},\mathbf{s}) &= \int_{-\pi}^{\pi}\overline{\delta\varepsilon_\theta}\, v\left.\frac{\partial\varepsilon_\theta}{\partial t}\right|_\beta \mathrm{d}\theta = \int_{-\pi}^{\pi}\overline{\delta\varepsilon_\theta}\, v\left(\frac{\partial\varepsilon_\theta}{\partial t} + \Omega\varepsilon'_\theta\right)\mathrm{d}\theta \\ &= \int_{-\pi}^{\pi}\overline{\delta\varepsilon_\theta}\, v\frac{\partial\varepsilon_\theta}{\partial t}\,\mathrm{d}\theta + \Omega\int_{-\pi}^{\pi}\overline{\delta\varepsilon_\theta} v\varepsilon'_\theta\,\mathrm{d}\theta. \end{aligned} \tag{A7}$$

Its discretized formulation leads to two matrices: the usual damping matrix **C** (symmetric positive) and an antisymmetric matrix $\Omega\mathbf{K}_c$.

A.1.5. *Discretized problem*

Equilibrium equations are written again with the viscous forces. With the discretization introduced, the natural frequency equation becomes

$$\begin{pmatrix} (\mathbf{K}+\Omega\mathbf{K}_c) & \Omega\mathbf{G} \\ \Omega\mathbf{G}^{\mathrm{T}} & \mathbf{M}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{U}\\ \mathbf{V}\end{pmatrix} = \mathrm{i}\omega\begin{pmatrix} -C & -\mathbf{M} \\ \mathbf{M} & \mathbf{0}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{U}\\ \mathbf{V}\end{pmatrix}. \tag{A8}$$

Denote by $\mathbf{B}_c$ the left side matrix and $\mathbf{A}_c$ the right side matrix. Because they are not symmetric anymore, the natural frequencies ω are now complex numbers.

A.2. PROJECTION FORMULAE WITH DAMPING

For the computation of the natural frequencies with damping and the mode shapes, it is possible to use the natural frequencies without damping as a starting value and to obtain the value of the natural frequencies ω^i with damping and the value of the modal shape vector $\mathbf{S}^i$ with damping by inverse iteration (see reference [17]).

The aim of this section is to develop the projection theorem of the force vector in order to get the solution.

Recall two classical results:

1. If ω^i is a natural frequency (complex) of equation (A8), then $-\overline{\omega^i}$ is a natural frequency of the adjoint problem.

   Note that $|B_c - i\omega^i \mathbf{A}_c|$ is the determinant of the matrix $(\mathbf{B}_c - i\omega^i \mathbf{A}_c)$. Suppose that ω^i is a natural frequency; then by definition

$$|\mathbf{B}_c - i\omega^i \mathbf{A}_c| = 0. \tag{A9}$$

   By transposition and conjugation, it is found that

$$|\mathbf{B}_c^* + \overline{i\omega^i}\mathbf{A}_c^*| = 0. \tag{A10}$$

2. Suppose now that the mode shapes of the following system $\delta\mathbf{S}^i$ are known:

$$\mathbf{B}_c^* \cdot \delta\mathbf{S}^i = -i\overline{\omega^i}\mathbf{A}_c^* \cdot \delta\mathbf{S}^i, \tag{A11}$$

   The projection theorem is

$$\omega^i \neq \omega^j \Rightarrow \delta\mathbf{S}^{i*} \cdot \mathbf{A}_c \cdot \mathbf{S}^j = 0. \tag{A12}$$

   To see this, choose $\mathbf{S}^j$ that verifies

$$\mathbf{B}_c \cdot \mathbf{S}^j = i\omega^j \mathbf{A}_c \cdot \mathbf{S}^j. \tag{A13}$$

   Multipling by $\delta\mathbf{S}^i$,

$$\delta\mathbf{S}^{i*} \cdot \mathbf{B}_c \cdot \mathbf{S}^j = i\omega^j \delta\mathbf{S}^{i*} \cdot \mathbf{A}_c \cdot \mathbf{S}^j. \tag{A14}$$

   Transpose and conjugate:

$$\mathbf{S}^{j*} \cdot \mathbf{B}_c^* \cdot \delta\mathbf{S}^i = -i\overline{\omega^j}\mathbf{S}^{j*} \cdot \mathbf{A}_c^* \cdot \delta\mathbf{S}^i. \tag{A15}$$

   By using the definition of $\delta\mathbf{S}^i$,

$$\mathbf{B}_c^* \cdot \delta\mathbf{S}^i = -i\overline{\omega^i}\mathbf{A}_c^* \cdot \mathbf{S}^i. \tag{A16}$$

   It is found that

$$-i\overline{\omega^i}\mathbf{S}^{j*} \cdot \mathbf{A}_c^* \cdot \delta\mathbf{S}^i = -i\overline{\omega^j}\mathbf{S}^{j*} \cdot \mathbf{A}_c^* \cdot \delta\mathbf{S}^i. \tag{A17}$$

This proves the result.

Coming back to the initial problem:

$$\mathbf{B}_c \cdot \mathbf{S} - \mathrm{i}\omega \mathbf{A}_c \cdot \mathbf{S} = \mathbf{F}, \tag{A18}$$

where $\mathbf{F}$ is the discretized vector representing the $\mathbf{q}_r(\omega, \theta)$ and $\mathbf{q}_\theta(\omega, \theta)$.

Suppose $\mathbf{S} = \sum_i a_i \mathbf{S}^i$; then by the projection theorem, it is found that

$$a_i = \frac{e_i}{\mathrm{i}(\omega^i - \omega)}, \tag{A19}$$

where, e_i, the projection of $\mathbf{F}$ on $\mathbf{S}^i$ is given by the formula

$$e_i = \frac{\delta \mathbf{S}^{i*} \cdot \mathbf{F}}{\delta \mathbf{S}^{i*} \cdot \mathbf{A}_c \cdot \mathbf{S}^i}. \tag{A20}$$

A.3. THE MODE SHAPES OF THE ADJOINT PROBLEM

In the above section, the mode shapes of the adjoint problem were assumed to be known. Giving a physical interpretation of these functions allows to construct them from the mode shapes of the direct problem. Rewrite $\mathbf{B}_c^* \cdot \delta \mathbf{S}^i = -\mathrm{i}\overline{\omega^i} \mathbf{A}_c^* \cdot \delta \mathbf{S}^i$:

$$\begin{pmatrix} (\mathbf{K} - \Omega \mathbf{K}_c) & \Omega \mathbf{G} \\ \Omega \mathbf{G}^{\mathrm{T}} & \mathbf{M} \end{pmatrix} \cdot \begin{pmatrix} \delta \mathbf{U} \\ \delta \mathbf{V} \end{pmatrix} = -\mathrm{i}\bar{\omega} \begin{pmatrix} -\mathbf{C} & \mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \delta \mathbf{U} \\ \delta \mathbf{V} \end{pmatrix}. \tag{A21}$$

Changing $\delta \mathbf{V}$ to $-\delta \mathbf{V}$ gives

$$\begin{pmatrix} (\mathbf{K} - \Omega \mathbf{K}_c) & -\Omega \mathbf{G} \\ \Omega \mathbf{G}^{\mathrm{T}} & -\mathbf{M} \end{pmatrix} \begin{pmatrix} \delta \mathbf{U} \\ -\delta \mathbf{V} \end{pmatrix} = -\mathrm{i}\bar{\omega} \begin{pmatrix} -\mathbf{C} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \delta \mathbf{U} \\ -\delta \mathbf{V} \end{pmatrix}. \tag{A22}$$

Multiplying the lower line by -1 gives

$$\begin{pmatrix} (\mathbf{K} - \Omega \mathbf{K}_c) & -\Omega \mathbf{G} \\ -\Omega \mathbf{G}^{\mathrm{T}} & \mathbf{M} \end{pmatrix} \begin{pmatrix} \delta \mathbf{U} \\ -\delta \mathbf{V} \end{pmatrix} = -\mathrm{i}\bar{\omega} \begin{pmatrix} -\mathbf{C} & -\mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \delta \mathbf{U} \\ -\delta \mathbf{V} \end{pmatrix}. \tag{A23}$$

It means that functions $\delta \mathbf{s}^i(\theta)$ can be obtained from the mode shapes of the reverse rotation problem, that is, the initial problem where Ω is changed in $-\Omega$. These functions are already known. From a mode shape of the direct problem, $\mathbf{s}^{-i}(\theta)$, with natural frequency $-\omega^i$, the following function is a mode shape of the reverse rotation problem:

$$\begin{aligned}\mathbf{u}(\theta) &= u_r^{-i}(\pi-\theta)\,\mathbf{e}_r(\theta) - u_\theta^{-i}(\pi-\theta)\,\mathbf{e}_\theta(\theta),\\ \mathbf{v}(\theta) &= v_r^{-i}(\pi-\theta)\,\mathbf{e}_r(\theta) - v_\theta^{-i}(\pi-\theta)\,\mathbf{e}_\theta(\theta).\end{aligned} \tag{A24}$$

The physical reason is explained in Figure A1. When an observer O_1 is looking at the tyre from the outside of the car, he sees a belt turning at rolling speed Ω. When an observer O_2 is looking at the same tyre from the inside of the car, he sees a belt turning at rolling speed $-\Omega$. For the second observer, Ω is changed to $-\Omega$, θ to $\pi-\theta$ and $\mathbf{e}_\theta$ is changed to $-e_\theta$. Of course, even if the description of the problem by the two observers are different, the dynamics of the problem is the same.

Finally, the mode shapes of the adjoint problem are obtained from the mode shapes of the reverse rotation problem when the relative speed is changed into its opposite. The modal shape function $\delta\mathbf{s}^i$ of the adjoint problem, associated with the natural pulsation $-\overline{\omega^i}$, is obtained from the modal shape function of the direct problem s^{-i} as follows:

$$\delta\mathbf{s}^i(\theta) = \begin{pmatrix} u_r^{-i}(\pi-\theta) \\ -u_\theta^{-i}(\pi-\theta) \\ -v_r^{-i}(\pi-\theta) \\ v_\theta^{-i}(\pi-\theta) \end{pmatrix}. \tag{A25}$$

Although this result can be proved with the equations of the "circular ring model", the reasoning with the two observers is independent of the model. It means that the above property is general and is not a special case for this model of rolling tyres.

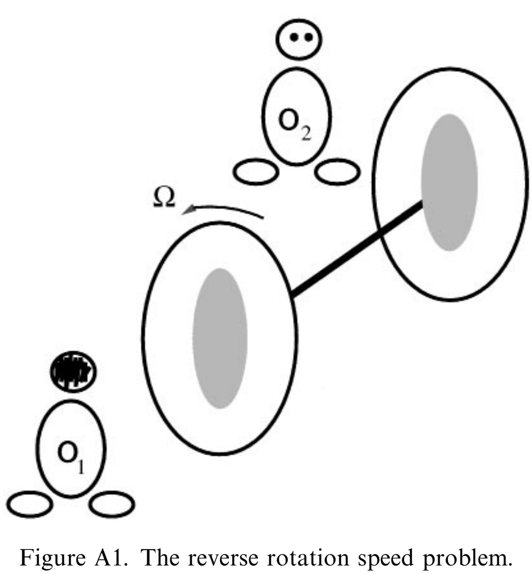

Figure A1. The reverse rotation speed problem.

Recall that equations of the adjoint problem are the same as equations of direct problem when there is no damping.

APPENDIX B: THE PROJECTION THEOREM IN THE FLOQUET THEORY

The solutions of equilibrium equations without source term are investigated. The following system is considered

$$\begin{aligned}
-\frac{1}{a^2}(D\chi_\theta)'' + \frac{K}{a}\varepsilon_\theta + \frac{1}{a}t_i\gamma_z' + (k_r + w_r(\beta))u_r \quad &+ \Omega\rho(\beta)(v_r' - v_\theta) &&= -\rho(\beta)\dot{v}_r,\\
-\frac{1}{a^2}(D\chi_\theta)' - \left(\frac{K}{a}\varepsilon_\theta\right)' + \frac{1}{a}t_i\gamma_z + (k_\theta + w_\theta(\beta))u_\theta \quad &+ \Omega\rho(\beta)(v_\theta' + v_r) &&= -\rho(\beta)\dot{v}_\theta,\\
-\Omega\rho(\beta)(u_r' - u_\theta) \quad &+ \rho(\beta)v_r &&= \rho(\beta)\dot{u}_r,\\
-\Omega\rho(\beta)(u_\theta' + u_r) \quad &+ \rho(\beta)v_\theta &&= \rho(\beta)\dot{u}_\theta,
\end{aligned} \tag{B1}$$

where $\beta = (\theta - \Omega t)$. Moreover, damping is added as in Appendix A.

For simplifications in notations, discretized formulation is used. Recall that the definition of the discretized state vector is

$$\mathbf{L}(t) = \begin{pmatrix} \mathbf{U}(t) \\ \mathbf{V}(t) \end{pmatrix} \tag{B2}$$

Recall also the two matrices

$$\mathbf{B}_c(t) = \begin{pmatrix} (\mathbf{K}(t) + \Omega\mathbf{K}_c) & \Omega\mathbf{G}(t) \\ -\Omega\mathbf{G}(t) & \mathbf{M}(t) \end{pmatrix} \tag{B3}$$

and

$$\mathbf{A}_c(t) = \begin{pmatrix} -\mathbf{C} & -\mathbf{M}(t) \\ \mathbf{M}(t) & \mathbf{0} \end{pmatrix}. \tag{B4}$$

B.1. CONSERVATION OF THE POISSON BRACKET

Suppose a solution of propagation equations without sources is known for $t \in [0, T]$:

$$\mathbf{B}_c(t)\cdot\mathbf{L} = \mathbf{A}_c(t)\cdot\dot{\mathbf{L}}. \tag{B5}$$

The initial state vector at $t = 0$ is arbitrary.

A discretized state vector $\delta\mathbf{L}(t)$ for $t \in [0, T]$ is chosen. Multiplying the system by the conjugate of this vector and integrating on $[0, T]$, one obtains

$$\int_0^T (\delta\mathbf{L}^* \cdot \mathbf{B}_c \cdot \mathbf{L} - \delta\mathbf{L}^* \cdot \mathbf{A}_c \cdot \dot{\mathbf{L}})\, \mathrm{d}t = 0. \tag{B6}$$

where $\delta\mathbf{L}^* = \overline{\delta\mathbf{L}}^T$

By transposing and conjugating this equation, are obtains

$$\int_0^T \left(\mathbf{L}^* \cdot \mathbf{B}_c^* \cdot \delta\mathbf{L} - \dot{\mathbf{L}}^* \cdot \mathbf{A}_c^* \cdot \delta\mathbf{L} \right) \mathrm{d}t = 0. \tag{B7}$$

Integrating by parts in time in the second term of the equation, one has

$$\int_0^T (\mathbf{L}^* \cdot \mathbf{B}_c^* \cdot \delta\mathbf{L} + \mathbf{L}^* \cdot \overset{\cdot}{\overbrace{\mathbf{A}_c^* \cdot \delta\mathbf{L}}})\, \mathrm{d}t = \mathbf{L}^*(T) \cdot \mathbf{A}_c^*(T) \cdot \delta\mathbf{L}(T) - \mathbf{L}^*(0) \cdot \mathbf{A}_c^*(0) \cdot \delta\mathbf{L}(0). \tag{B8}$$

Suppose now that $\delta\mathbf{L}(t)$ is chosen in order to have

$$\mathbf{B}_c^* \cdot \delta\mathbf{L} = -\overset{\cdot}{\overbrace{A_c^* \cdot \delta\mathbf{L}}} \tag{B9}$$

then, the following property holds:

$$\mathbf{L}^*(T) \cdot \mathbf{A}_c^*(T) \cdot \delta\mathbf{L}(T) = \mathbf{L}^*(0) \cdot \mathbf{A}_c^*(0) \cdot \delta\mathbf{L}(0). \tag{B10}$$

By transposing and conjugating this equation, one obtains the conservation of Poisson's bracket:

$$\delta\mathbf{L}^*(T) \cdot \mathbf{A}_c(T) \cdot \mathbf{L}(T) = \delta\mathbf{L}^*(0) \cdot \mathbf{A}_c(0) \cdot \mathbf{L}(0). \tag{B11}$$

B.2. THE ADJOINT PROPAGATION OPERATOR IN FLOQUET THEORY

The function $\delta\mathbf{L}(t)$ was assumed to follow the equation

$$\mathbf{B}_c^* \cdot \delta\mathbf{L} = -\overset{\cdot}{\overbrace{A_c^* \cdot \delta\mathbf{L}}}. \tag{B12}$$

As in Appendix A, it will be shown that this equation is related to the reverse rotation problem. Express the value of the above expressions:

$$\mathbf{B}_c^* \cdot \delta\mathbf{L} = \begin{pmatrix} (\mathbf{K} - \Omega\mathbf{K}_c) & -\Omega\mathbf{G}^*(t) \\ \Omega\mathbf{G}^*(t) & \mathbf{M}(t) \end{pmatrix} \begin{pmatrix} \delta\mathbf{U} \\ \delta\mathbf{V} \end{pmatrix} \tag{B13}$$

and

$$-\overbrace{A_c^* \cdot \delta \mathbf{L}}^{\cdot} = -\begin{pmatrix} -\mathbf{C} & \mathbf{M}(t) \\ -\mathbf{M}(t) & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} \delta\dot{\mathbf{U}} \\ \delta\dot{\mathbf{V}} \end{pmatrix} - \begin{pmatrix} \mathbf{0} & \dot{\mathbf{M}}(t) \\ -\dot{\mathbf{M}}(t) & \mathbf{0} \end{pmatrix} \begin{pmatrix} \delta \mathbf{U} \\ \delta \mathbf{V} \end{pmatrix} \cdot \quad \text{(B14)}$$

Calculate now the value of the non-diagonal terms $\Omega \mathbf{G}^*(t) - \dot{\mathbf{M}}(t)$. Using the notation introduced in Appendix A, these terms are

$$\Omega \int_{-\pi}^{\pi} \rho(\beta) N_r^i (N_r^{j\prime} - N_\theta^j)\,\mathrm{d}\theta + \Omega \int_{-\pi}^{\pi} \rho(\beta) N_\theta^i (N_\theta^{j\prime} + N_r^j)\,\mathrm{d}\theta$$
$$- \int_{-\pi}^{\pi} \dot{\rho}(\beta) N_r^j N_r^i\,\mathrm{d}\theta - \int_{-\pi}^{\pi} \dot{\rho}(\beta) N_\theta^j N_\theta^i \mathrm{d}\theta. \quad \text{(B15)}$$

Reordering in N_r^j and N_θ^j, these terms becomes

$$\Omega \int_{-\pi}^{\pi} \rho(\beta)(N_r^i N_r^{j\prime} + N_\theta^i N_r^j)\,\mathrm{d}\theta + \Omega \int_{-\pi}^{\pi} \rho(\beta)(N_\theta^i N_\theta^{j\prime} - N_r^i N_\theta^j)\,\mathrm{d}\theta$$

$$- \int_{-\pi}^{\pi} \dot{\rho}(\beta) N_r^j N_r^i\,\mathrm{d}\theta - \int_{-\pi}^{\pi} \dot{\rho}(\beta) N_\theta^j N_\theta^i\,\mathrm{d}\theta. \quad \text{(B16)}$$

Integrating by parts in θ, these terms become:

$$-\Omega \int_{-\pi}^{\pi} \rho(\beta) N_r^j (N_r^{i\prime} - N_\theta^i)\,\mathrm{d}\theta - \Omega \int_{-\pi}^{\pi} \rho(\beta) N_\theta^j (N_\theta^{i\prime} + N_r^i)\,\mathrm{d}\theta$$

$$- \int_{-\pi}^{\pi} (\dot{\rho}(\beta) + \Omega\rho'(\beta)) N_r^j N_r^i\,\mathrm{d}\theta - \int_{-\pi}^{\pi} (\dot{\rho}(\beta) + \Omega\rho'(\beta)) N_\theta^j N_\theta^i\,\mathrm{d}\theta. \quad \text{(B17)}$$

Remembering the equation of conservation of the mass $\dot{\rho} + \Omega\rho' = 0$, it is found that

$$\Omega \int_{-\pi}^{\pi} \rho(\beta) N_r^i (N_r^{j\prime} - N_\theta^j)\,\mathrm{d}\theta + \Omega \int_{-\pi}^{\pi} \rho(\beta) N_\theta^i (N_\theta^{j\prime} + N_r^j)\,\mathrm{d}\theta - \int_{-\pi}^{\pi} \dot{\rho}(\beta) N_r^j N_r^i\,\mathrm{d}\theta$$

$$- \int_{-\pi}^{\pi} \dot{\rho}(\beta) N_\theta^j N_\theta^i\,\mathrm{d}\theta = -\Omega \int_{-\pi}^{\pi} \rho(\beta) N_r^j (N_r^{i\prime} - N_\theta^i)\,\mathrm{d}\theta$$

$$-\Omega \int_{-\pi}^{\pi} \rho(\beta) N_\theta^j (N_\theta^{i\prime} + N_r^i)\,\mathrm{d}\theta, \quad \text{(B18)}$$

that is

$$\Omega \mathbf{G}^*(t) - \dot{\mathbf{M}}(t) = -\Omega \mathbf{G}(t). \quad \text{(B19)}$$

Taking this into account, it is found that the vector $\delta\mathbf{L}$ follows

$$\begin{pmatrix} (\mathbf{K}(t) - \Omega\mathbf{K}_c) & \Omega\mathbf{G}(t) \\ -\Omega\mathbf{G}(t) & \mathbf{M}(t) \end{pmatrix} \begin{pmatrix} \delta\mathbf{U} \\ \delta\mathbf{V} \end{pmatrix} = -\begin{pmatrix} -\mathbf{C} & \mathbf{M}(t) \\ -\mathbf{M}(t) & \mathbf{0} \end{pmatrix} \begin{pmatrix} \delta\dot{\mathbf{U}} \\ \delta\dot{\mathbf{V}} \end{pmatrix}. \tag{B20}$$

Introduce now the same transformation as in Appendix A. Changing $\delta\mathbf{V}$ to $-\delta\mathbf{V}$ gives

$$\begin{pmatrix} (\mathbf{K}(t) - \Omega\mathbf{K}_c) & -\Omega\mathbf{G}(t) \\ -\Omega\mathbf{G}(t) & -\mathbf{M}(t) \end{pmatrix} \begin{pmatrix} \delta\mathbf{U} \\ -\delta\mathbf{V} \end{pmatrix} = -\begin{pmatrix} -\mathbf{C} & -\mathbf{M}(t) \\ -\mathbf{M}(t) & \mathbf{0} \end{pmatrix} \begin{pmatrix} \delta\dot{\mathbf{U}} \\ -\delta\dot{\mathbf{V}} \end{pmatrix}. \tag{B21}$$

Multiplying the lower line by -1 gives

$$\begin{pmatrix} (\mathbf{K}(t) - \Omega\mathbf{K}_c) & -\Omega\mathbf{G}(t) \\ \Omega\mathbf{G}(t) & \mathbf{M}(t) \end{pmatrix} \begin{pmatrix} \delta\mathbf{U} \\ -\delta\mathbf{V} \end{pmatrix} = -\begin{pmatrix} -\mathbf{C} & -\mathbf{M}(t) \\ \mathbf{M}(t) & \mathbf{0} \end{pmatrix} \begin{pmatrix} \delta\dot{\mathbf{U}} \\ -\delta\dot{\mathbf{V}} \end{pmatrix}. \tag{B22}$$

Finally change t to $t' = T - t$, and as a consequence $\partial/\partial t$ to $-\partial/\partial t'$:

$$\begin{pmatrix} (\mathbf{K}(t') - \Omega\mathbf{K}_c) & -\Omega\mathbf{G}(t') \\ \Omega\mathbf{G}(t') & \mathbf{M}(t') \end{pmatrix} \begin{pmatrix} \delta\mathbf{U} \\ -\delta\mathbf{V} \end{pmatrix} = \begin{pmatrix} -\mathbf{C} & -\mathbf{M}(t') \\ \mathbf{M}(t') & \mathbf{0} \end{pmatrix} \frac{\partial}{\partial t'} \begin{pmatrix} \delta\mathbf{U} \\ -\delta\mathbf{V} \end{pmatrix}. \tag{B23}$$

This problem is the initial problem, except that Ω must be changed to $-\Omega$, it is called the reverse rotation problem.

The mode shapes of the reverse rotation problem are already known. From a mode shape of the direct problem, $\mathbf{l}(t', \theta)$, t' in $[0, T]$, the following function is a solution of the reverse rotation problem:

$$\delta\mathbf{u}(t', \theta) = u_r(t', \pi - \theta)\,\mathbf{e}_r(\theta) - u_\theta(t', \pi - \theta)\,\mathbf{e}_\theta(\theta),$$

$$\delta\mathbf{v}(t', \theta) = v_r(t', \pi - \theta)\,\mathbf{e}_r(\theta) - v_\theta(t', \pi - \theta)\,\mathbf{e}_\theta(\theta). \tag{B24}$$

The physical reason for this is the same as in Appendix A (see Figure 5).

Finally, a solution of the adjoint problem is obtained from a solution of the reverse rotation problem when the relative speed revised, and the time t' is changed to $T - t$:

$$\delta\mathbf{L}(t, \theta) = \begin{pmatrix} u_r(T - t, \pi - \theta) \\ -u_\theta(T - t, \pi - \theta) \\ -v_r(T - t, \pi - \theta) \\ v_\theta(T - t, \pi - \theta) \end{pmatrix}. \tag{B25}$$

B.3. FLOQUET DECOMPOSITION OF THE ADJOINT PROBLEM

In the last section, the adjoint problem was introduced. In this section, its Floquet coefficients and its eigenfunctions that are investigated.

Denote by P^*, the linear operator that integrates the adjoint equations. Knowing an initial value of a Cauchy type at time $t = 0$, $\delta\mathbf{l}(0, \theta)$, P^* gives the solution at time $t = T$ as a linear form of the initial condition:

$$\delta\mathbf{l}(T, \theta) = P^*[\delta\mathbf{l}(0, \theta)]. \tag{B26}$$

It will be shown that the mode shapes of the adjoint problem and the Floquet coefficients can be calculated from those of the direct problem. Suppose that functions $\mathbf{l}^i(t, \theta)$ of $t \in [0, T]$, and numbers λ^i are known:

$$P[\mathbf{l}^i(0, \theta)] = \lambda^i \mathbf{l}^i(0, \theta). \tag{B27}$$

Setting

$$\delta\mathbf{l}^i(t, \theta) = \begin{pmatrix} u_r^i(T - t, \pi - \theta) \\ -u_\theta^i(T - t, \pi - \theta) \\ -v_r^i(T - t, \pi - \theta) \\ v_\theta(T - t, \pi - \theta) \end{pmatrix}, \tag{B28}$$

a solution of the adjoint problem for $t \in [0, T]$ is obtained. Moreover, this solution verifies

$$P^*[\delta\mathbf{l}^i(0, \theta)] = \frac{1}{\lambda^i} \delta\mathbf{l}^i(0, \theta). \tag{B29}$$

It means that the eigenfunctions of the adjoint problem can be deduced from the eigenfunctions of the direct problem, and that the Floquet coefficients of the adjoint problem are the inverses of the Floquet coefficients of the direct problem.

This result comes from the property of the function $\mathbf{l}^i(t, \theta)$:

$$\mathbf{l}^i(T, \theta) = \lambda^i \mathbf{l}^i(0, \theta). \tag{B30}$$

When the relative speed is revised, and the time is changed to $T - t$, are obtains

$$\delta\mathbf{l}^i(0, \theta) = \lambda^i \delta\mathbf{l}^i(T, \theta). \tag{B31}$$

Remembering that $\delta\mathbf{l}^i(T, \theta) = P^*[\delta\mathbf{l}^i(0, \theta)]$, the proof of the previous result is obtained.

B.4. PROJECTION FORMULAE

B.4.1. General case

Suppose that the set of mode shapes $\mathbf{l}_0^i(\theta)$ and the values λ^i have been calculated. By propagation $\overline{\mathbf{l}_0^i}$ becomes $\overline{P[\mathbf{l}_0^i(\theta)]}$ and is also an eigenvector of P with eigenvalue

$\overline{\lambda^i}$. By convention, set $\mathbf{l}_0^{-i} = \overline{\mathbf{l}_0^i}$ and $\lambda^{-i} = \overline{\lambda^i}$. Suppose also that from these values, the set of adjoint modal shapes $\delta\mathbf{l}_0^j(\theta)$ and the numbers $1/\lambda^j$ have been calculated.

Choose two modal shapes $\mathbf{l}_0^i(\theta)$ and $\delta\mathbf{l}_0^{-j}(\theta)$. The first one verifies

$$P[\mathbf{l}^i(0,\theta)] = \lambda^i \mathbf{l}^i(0,\theta) \tag{B32}$$

and the second one

$$P^*[\delta\mathbf{l}_0^{-j}(\theta)] = \frac{1}{\overline{\lambda^j}}\,\delta\mathbf{l}_0^{-j}(\theta). \tag{B33}$$

Apply now these properties to relation (B11):

$$\frac{1}{\lambda^j}\,\delta\mathbf{L}_0^{-j*}\cdot\mathbf{A}_c(T)\cdot\lambda^i\mathbf{L}_0^i = \delta\mathbf{L}_0^{-j*}\cdot\mathbf{A}_c(0)\cdot\mathbf{L}_0^i. \tag{B34}$$

This shows the projection formulae

$$\lambda^i \neq \lambda^j \Rightarrow \delta\mathbf{L}_0^{-j*}\cdot\mathbf{A}_c(0)\cdot\mathbf{L}_0^i = 0. \tag{B35}$$

B.4.2. Special case without damping

When the damping is neglected, the adjoint problem is described by the following equations

$$\begin{pmatrix} \mathbf{K}(t) & \Omega\mathbf{G}(t) \\ -\Omega\mathbf{G}(t) & \mathbf{M}(t) \end{pmatrix}\begin{pmatrix} \delta\mathbf{U} \\ \delta\mathbf{V} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & -\mathbf{M}(t) \\ \mathbf{M}(t) & \mathbf{0} \end{pmatrix}\begin{pmatrix} \delta\dot{\mathbf{U}} \\ \delta\dot{\mathbf{V}} \end{pmatrix}. \tag{B36}$$

These equations are the same as the direct problem. This means that a mode shape function of the adjoint problem $\delta\mathbf{l}_0^j(\theta)$ is in fact a mode shape function of the direct problem with Floquet coefficient $1/\lambda^j = \lambda^{-j}$. The last equality is valid because without dissipation, the Floquet coefficient modulus is one.

Remembering that $\delta\mathbf{l}_0^{-j*} = \delta\mathbf{l}_0^{jT}$, the projection formulae becomes

$$\lambda^i \neq \lambda^j \Rightarrow \mathbf{L}_0^{jT}\cdot\mathbf{A}(0)\cdot\mathbf{L}_0^i = 0. \tag{B37}$$

APPENDIX C. THE DIAGONALIZATION IN THE FLOQUET THEORY

For the diagonalization of the operator of propagation P, it is not possible to use the Lanczos algorithm directly. Nevertheless, a similar algorithm can be used:

- Choose a random vector $\mathbf{L}_0$, calculate the vector $\delta\mathbf{L}_0$ with the projection formulae of Appendix B.
- Use this vector as an initial condition of a Cauchy type in a time integration algorithm. The most used algorithm is the Hilber Hughes Taylor algorithm (HHT). The vector $\tilde{\mathbf{L}}_1 = P[\mathbf{L}_0]$ is then obtained.

- Construct from the vector $\tilde{\mathbf{L}}_1$ the second vector $\mathbf{L}_1$ using the formula

$$\mathbf{L}_1 = \tilde{\mathbf{L}}_1 - p_{1,1}\mathbf{L}_0 \tag{C1}$$

$p_{1,1}$ is chosen so that

$$\delta\mathbf{L}_0^* \cdot \mathbf{A}_c \cdot \mathbf{L}_1 = 0. \tag{C2}$$

It is found that $p_{1,1}$ is given by

$$(\delta\mathbf{L}_0^* \cdot \mathbf{A}_c \cdot \mathbf{L}_0)p_{1,1} = \delta\mathbf{L}_0^* \cdot \mathbf{A}_c \cdot \tilde{\mathbf{L}}_1 \tag{C3}$$

Finally, calculate the vector $\delta\mathbf{L}_1$ with the formulae of Appendix B.
- More generally, suppose that $\mathbf{L}_0, \dots, \mathbf{L}_k$ have been constructed. Suppose also that the vectors $\delta\mathbf{L}_0, \dots, \delta\mathbf{L}_k$ are known and verify:

$$\forall m \neq n \quad \delta\mathbf{L}_m^* \cdot \mathbf{A}_c \cdot \mathbf{L}_n = 0. \tag{C4}$$

- Use the last vector $\mathbf{L}_k$ as a initial condition of a Cauchy type in a time. The vector $\tilde{\mathbf{L}}_{k+1} = P[\mathbf{L}_k]$ is then obtained.
- Construct from the vector $\tilde{\mathbf{L}}_{k+1}$ the new vector $\mathbf{L}_{k+1}$ using the formula

$$\mathbf{L}_{k+1} = \tilde{\mathbf{L}}_{k+1} - \sum_{m=1}^{k+1} p_{m,k+1}\mathbf{L}_{m-1} \tag{C5}$$

$p_{m,k+1}$ is chosen so that

$$\delta\mathbf{L}_{m-1}^* \cdot \mathbf{A}_c \cdot \mathbf{L}_{k+1} = 0. \tag{C6}$$

It is found that $p_{m,k+1}$ is given by

$$(\delta\mathbf{L}_{m-1}^* \cdot \mathbf{A}_c \cdot \mathbf{L}_{m-1})p_{m,k+1} = \delta\mathbf{L}_{m-1}^* \cdot \mathbf{A}_c \cdot \tilde{\mathbf{L}}_{k+1}. \tag{C7}$$

The difference from the Lanczos algorithm is that the new vector must be a combination of all the precedent vectors and not of the two precedent vectors.
- Once vectors $\mathbf{L}_0, \dots, \mathbf{L}_\mathrm{K}$ are known, it is possible to diagonalize the matrix P on this reduced subspace. On this subspace, P is a Hessemberg superior matrix whose value is $[p_{i,j}]$, plus a rest $\mathbf{L}_{K+1}$:

$$P \cdot (\mathbf{L}_0, \dots, L_\mathrm{K}) = (\tilde{\mathbf{L}}_1, \dots, \tilde{\mathbf{L}}_{K+1}) = (\mathbf{L}_0, \dots, \mathbf{L}_\mathrm{K})$$

$$\times \begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} & p_{1,4} & \dots \\ 1 & p_{2,2} & p_{2,3} & p_{2,4} & \dots \\ 0 & 1 & p_{3,3} & p_{3,4} & \dots \\ 0 & 0 & 1 & p_{4,4} & \dots \\ 0 & 0 & 0 & 1 & \dots \end{pmatrix} + (0, \dots, \mathbf{L}_{K+1}). \tag{C8}$$

- This procedure allows to find the mode shapes that are in the subspace $\mathbf{L}_0, \ldots, \mathbf{L}_K$. Examine this subspace: it contains the Krylov sequence of $\mathbf{L}_0$, that is, the successive vectors $\mathbf{L}_0, P[\mathbf{L}_0], P^2[\mathbf{L}_0], \ldots, P^K[\mathbf{L}_0]$. This sequence is known to contain the mode shapes of P associated with the Floquet coefficients of largest modulus. These are those that are wanted. Moreover, the following property is true by construction

$$\delta \mathbf{L}_m^* \cdot \mathbf{A}_c \cdot \mathbf{L}_{K+1} = 0. \tag{C9}$$

If a mode shape $\mathbf{L}^i$ is in the subspace, then

$$\delta \mathbf{L}^{i*} \cdot \mathbf{A}_c \cdot \mathbf{L}_{K+1} = 0. \tag{C10}$$

It means that the rest L_{K+1} is automatically purged from the converged mode shapes.

This algorithm should be compared with the reasoning used in reference [7]. If the damping is strong, only the waves emitted at the last instants have a significant amplitude. In particular, in the equation of propagation with source term, it means that the initial condition has no influence on the solution after a certain time of integration. In this case, the solution of the problem is the special solution with zero initial value, after a long time of integration.

In this appendix, the algorithm also proposes to integrate the equations. The advantages are that the excitation force can be unknown, and that when a mode shape is contained in the Krylov sequence, it is automatically detected by the algorithm.